HW Solutions, 2.1

2.1, 22 Show that if the columns of B are linearly dependent, then so are the columns of AB.

SOLUTION 1: Let $B = [\mathbf{b}_1, \ldots \mathbf{b}_k]$. If the columns of B are linearly dependent, then there is a set of constants $c_1, \ldots, c_k$, not all zero, so that

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k = 0$$

We note that AB is formed as the matrix:

$$AB = A[\mathbf{b}_1, \ldots \mathbf{b}_k] = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots A\mathbf{b}_k]$$

so that these are the columns of AB. Now, if we multiply the previous equation by A, we must have constants $c_1, \ldots, c_k$, not all zero, so that:

$$c_1 A\mathbf{b}_1 + c_2 A\mathbf{b}_2 + \cdots + c_k A\mathbf{b}_k = 0$$

Therefore, the columns of AB are linearly dependent as well. And, as we noted in class, this implies that the dependence relation in the columns of B are **exactly the same** as in AB.

Alternative Solution: We could use a theorem about linear dependence instead of the definition. In that case, if the columns of B are linearly dependent, there is a nontrivial solution $\mathbf{x}$ so that

$$B\mathbf{x} = \mathbf{0}.$$

Multiply both side by A, and we see that $\mathbf{x}$ is a nontrivial solution to:

$$(AB)\mathbf{x} = \mathbf{0}$$

so the columns of AB are linearly dependent.

2.1, 24 Suppose that $AD = I_m$. Show that for any $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

SOLUTION: We show that $\mathbf{x} = D\mathbf{b}$:

$$A\mathbf{x} = A(D\mathbf{b}) = (AD)\mathbf{b} = I\mathbf{b} = \mathbf{b}$$