Misc. Review Problems, Exam 2
Math 338
Fall 2007

As usual, these questions are here to give you a chance to review problems out of context, and not necessarily exhaustive. You should be sure that you understand the quizzes and homework problems that have been assigned. You will be given a list of the distributions that we have looked at- Note that some were missing from Ch 5, all from Ch 6 will be included.
Specific Sections:
4.3-4.8, 5.1-5.7, 6.1-6.6 (omit 6.4)

1. Draw a flow chart that relates all the different distributions (if such a connection exists). For example, the geometric distribution is a special case of ??? (using what parameters)?

   We'll look at what you have in class on Thursday.

2. Exercise 5.19: Compute the mean and variance of the negative binomial distribution (the technique is what is important here- factor out what we need and manipulate the sum).

$$b^*(x;k,\theta) = \binom{x-1}{k-1}\theta^k(1-\theta)^{x-k}, \qquad x = k, k+1, k+2, \ldots$$

   The problem suggests that we use $E(X)$ and $E(X(X+1))$. We notice then that

$$E(X^2) = E(X(X+1)) - E(X)$$

   from which we compute $\sigma^2 = E(X^2) - (E(X))^2$.
   Now, we write $E(X)$ and note that we want to factor k/θ out, and the remaining sum should sum to 1:

$$E(X) = \sum_{x=k}^{\infty} \frac{x \cdot (x-1)!}{(k-1)!(x-k)!}\theta^k(1-\theta)^{x-k}$$

   We'll go ahead and factor out what should be the mean, k/θ, and see if the remaining stuff sums to 1.

$$E(X) = \frac{k}{\theta}\sum_{x=k}^{\infty} \frac{x!}{k!(x-k)!}\theta^{k+1}(1-\theta)^{x-k}$$

   To see that the series sums to 1, we could make the substitutions:

$$\begin{array}{l} x = y-1 \\ k = n-1 \end{array} \quad \Rightarrow \quad \sum_{y=n}^{\infty}\binom{y-1}{n-1}\theta^n(1-\theta)^{y-n} = 1$$

Therefore, $E(X) = k/\theta$. The next part continues using the same techniques. As suggested by the problem, we compute $E(X(X+1))$:

$$\sum_{x=k}^{\infty} \frac{(x+1)\cdot x \cdot (x-1)!}{(k-1)!(x-k)!}\theta^k (1-\theta)^{x-k} = \sum_{x=k}^{\infty} \frac{(x+1)!}{(k-1)!(x-k)!}\theta^k (1-\theta)^{x-k}$$

Since the numerator is $(x+1)!$, I would like to have $(k+1)!$ in the denominator to pull off the desired factorization (to get the series to sum to 1). If I had the $(k+1)!$ in the denominator, however, I would also need the power of θ to be $k+2$. We can get all of these: Divide the series by $k(k+1)$ and multiply it by θ^2:

$$\frac{k(k+1)}{\theta^2} \sum_{x=k}^{\infty} \frac{(x+1)!}{(k+1)!(x-k)!}\theta^{k+2}(1-\theta)^{x-k}$$

The series will sum to 1 (that's how we designed it). To see it formally, make the substitutions:

$$\begin{array}{c} x+1 = y-1 \\ k+1 = n-1 \end{array} \quad \Rightarrow \quad \sum_{y=n}^{\infty} \binom{y-1}{n-1} \theta^n (1-\theta)^{y-n} = 1$$

Now we have $E(X(X+1)) = k(k+1)/\theta^2$, from which we can compute $E(X^2)$:

$$E(X^2) = E(X(X+1)) - E(X) = \frac{k(k+1) - k\theta}{\theta^2}$$

And now the variance:

$$\sigma^2 = E(X^2) - (E(X))^2 = \frac{k^2 + k - k\theta}{\theta^2} - \frac{k^2}{\theta^2} = \frac{k}{\theta}\left(\frac{1}{\theta} - 1\right)$$

3. The distribution of some weeds across my yard looks like it might be random. Suppose there are 75 dandelions in my rectangular yard, and I divide my yard into a 10×10 grid.

   The following is a summary of what I got- Is the distribution random?

Number of Weeds	0	1	2	3	4
Number of squares	47	36	13	3	1
Theory Says	47.23	35.43	13.29	3.32	0.62

   By the way, if you x is the first row and y is the second, what should you get with $\sum x_i y_i$? With $\sum y_i$?

   Fill in the table (I added the row above) with the theoretical probabilities from the Poisson distribution,

$$\lambda = \frac{75 \text{ weeds}}{100 \text{ squares}} \qquad \Rightarrow \qquad 100\left(\frac{\lambda^x \mathrm{e}^{-\lambda}}{x!}\right)$$

   The sum $\sum_i y_i$ is the number of weeds, 75, and $\sum y_i$ is the total number of squares.

4. Suppose that experience has shown that the length of time X required to conduct a periodic maintenance check on an ipod follows a Gamma distribution with $\alpha = 3.1$ and $\beta = 2$. A new maintenance worker takes 22.5 minutes to check the machine.

   Use Chebyshev's inequality to see if this length of time agrees with our past experience.

   Using Chebyshev's inequality with mean $(3.1)(2) = 6.2$ and standard deviation $\sqrt{\alpha\beta^2} = \sqrt{12.4} \approx 3.52$, we're wondering about the probability that the rv is 22.5.

   So we need to determine how many standard deviations 22.5 is from the mean:

$$\mu + k\sigma = 22.5 \quad \Rightarrow \quad 6.2 + k \cdot 3.52 = 22.5 \quad \Rightarrow \quad k \approx 4.63$$

   So what is the probability that we are 4.63 standard deviations away from the mean? Very small, since

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{4.63^2} \approx 0.0466$$

   We would conclude that this is probably due to the fact that a new worker will be slower.

5. Find the moment generating function for the exponential distribution. Use it to verify the mean and variance.

   SOLUTION:

$$M(t) = E(\mathrm{e}^{tX}) = \frac{1}{\theta}\int_0^\infty \mathrm{e}^{tx}\mathrm{e}^{-x/\theta}\,dx = \frac{1}{\theta}\int_0^\infty \mathrm{e}^{-x(\frac{1}{\theta}-t)}\,dx = \frac{-1}{1-\theta t} \cdot \mathrm{e}^{-x(\frac{1}{\theta}-t)}\Big|_0^\infty$$

   The limit converges as long as $1 - \theta t > 0$, or for $t < \frac{1}{\theta}$. Notice that this is fine, since we are typically interested in $M^{(n)}(0)$. Continuing then, we've shown that the MGF is:

$$M(t) = \frac{1}{1-\theta t} \qquad M'(t) = \frac{\theta}{(1-\theta t)^2} \qquad M''(t) = \frac{2\theta^2}{(1-\theta t)^3}$$

   Therefore,

$$\mu = E(X) = M'(0) = \theta$$

   and

$$\sigma^2 = E(X^2) - (E(X))^2 = 2\theta^2 - \theta^2 = \theta^2$$

6. Given that the moment generating function of the normal distribution is:

$$M(t) = \mathrm{e}^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

   if X is a normally distributed rv with mean 4 and standard deviation 2, what is the moment generating function for $Y = -3X + 4$? What is the distribution of Y (i.e., can you recognize the mgf?)

The main idea here is to use the relationship between the MGF of X and the MGF of $aX+b$ (Theorem 4.10, Page 146). That is, if X has MGF of $M(t)$ and $Y = aX+b$, then:

$$M_Y(t) = E(\mathrm{e}^{(aX+b)t}) = E(\mathrm{e}^{X(at)}\mathrm{e}^{bt}) = \mathrm{e}^{bt}M(at)$$

Therefore, in this case,

$$M_Y(t) = \mathrm{e}^{4t}M(-3t)$$

Now, we know what $M(t)$ is,

$$M(t) = \mathrm{e}^{4t+2t^2} \quad \Rightarrow \quad M(-3t) = \mathrm{e}^{-12t+18t^2}$$

Therefore, the distribution of Y has the mgf:

$$M_Y(t) = \mathrm{e}^{4t}\mathrm{e}^{-12t+18t^2} = \mathrm{e}^{-8t+18t^2}$$

This is another normal distribution with mean -8 (the original distribution had a mean of 4), and a standard deviation of 6.

7. A company manufactures and bottles juice in 16 oz containers. The amount actually dispensed has been observed to be approximately normal with mean 16 and standard deviation 1 oz. What proportion of bottles will have more than 17 oz dispensed into them?

$$P(X \geq 17) = P\left(\frac{X-\mu}{\sigma} \geq 1\right) = P(Z \geq 1)$$

Using Table III, this is $\frac{1}{2} - P(0 \leq Z \leq 1) = 0.1587$

So, there is about a 16% chance of a bottle getting overfull.

8. Let $f(x;\theta)$ be the Bernoulli distribution. Find the mean and variance directly.

$$\mu = 0 \cdot (1-\theta) + 1 \cdot \theta = \theta$$

$$E(X^2) = 0^2 \cdot (1-\theta) + 1^2 \cdot \theta = \theta \qquad \Rightarrow \qquad \sigma^2 = \theta - \theta^2 = \theta(1-\theta)$$

Let X_i each be a Bernoulli rv with the same parameter θ, and they are independent. If $Y = X_1 + X_2 + X_3 + \cdots + X_n$, compute the mean and variance of Y.

This was Exercise 5.6:

Now, using the properties of the expected value and variance, if $Y = X_1 + X + 2 + \cdots + X_n$, where each of these are Bernoulli with the same parameter and they are independent,

$$E(Y) = E(X_1) + \cdots + E(X_n) = \theta + \theta + \cdots + \theta = n\theta$$

We compute the variance, and use the fact that X_i, X_j are independent for $i \neq j$:

$$E(Y^2) = E(X_1^2) + \cdots E(X_n^2) = \sigma_{x_1}^2 + \cdots + \sigma_{x_n}^2 = n\theta(1-\theta)$$

9. Suppose a large stockpile of used pumps contains 20% that are in need of repair. A maintenance worker is sent to the stockpile with three repair kits. He selects pumps at random and tests them one at a time. If the pump works, he sets it aside for future work. If not, he repairs it. Suppose that it takes 10 minutes to test a pump that is in working order, and 30 minutes to test and repair a pump that does not work. Find the mean and variance of the total time it takes the maintenance worker to use the three repair kits.

   Let Y be the number of the trial on which the third bad pump is found. Think of this as the number of trials until the third "success", so Y is a negative binomial distribution with $\theta = 0.2$.

   To find the mean and variance of the time it takes, we need to know how many trials it took. If it takes Y trials to use up the three pumps, then the total time overall would be 10 minutes per Y trials, plus an additional 20 minutes for the three bad pumps, or

$$T = 10Y + 60$$

   To find the mean and variance of T, we get the mean and variance of Y (use the formulas):

$$\mu_Y = \frac{3}{0.2} = 15 \quad \Rightarrow \quad \sigma_Y^2 = 60$$

   Therefore,

$$\mu_T = 10\mu_Y + 60 = 210 \qquad \sigma_T^2 = 10^2\sigma_Y^2 = 6000$$

   The total time is about 3.5 hours, with a standard deviation of about 1.3 hours.

10. Suppose X has a normal distribution, $n(x; 2, 3)$. Write $P(X \leq 4)$ so that you can look up the needed probabilities in Table III.

$$P(X \leq 4) = P\left(Z \leq \frac{2}{3}\right) = \frac{1}{2} + P\left(0 \leq Z \leq \frac{2}{3}\right) = 0.5 + 0.2486 = 0.7486$$

   (Note the fraction was rounded to 0.67 for the Table)

11. Find the moment generating function for the rv X, if X has the continuous pdf $f(x) = 1$ for $0 < x < 1$, zero elsewhere.

$$E\left(\mathrm{e}^{tX}\right) = \int_0^1 \mathrm{e}^{tx}\, dx = \frac{1}{t}\mathrm{e}^{tx}\Big|_0^1 = \frac{\mathrm{e}^t - 1}{t}$$

   Use it to find the mean and variance.

$$M'(t) = \frac{e^t(t-1)+1}{t^2} \qquad M''(t) = \frac{e^t(t^2-2t+2)-2}{t^3}$$

Yes, it's a pain, but use l'Hospital's rule to compute the limit as $t \to 0$. Doing this, you get:

$$E(X) = \frac{1}{2} \qquad E(X^2) = \frac{1}{3} \qquad \sigma^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

12. Using the definition of Γ, compute $Gamma(5)$. Hint: You might want to use a table for integration by parts, and the Lemma we had in class about polynomials and exponentials.

    Using the table to integrate $\Gamma(5) = \int_0^\infty y^4 e^{-y}\, dy$, we get:

sign	du	v
+	y^4	e^{-y}
−	$4y^3$	$-e^{-y}$
+	$12y^2$	e^{-y}
−	$24y$	$-e^{-y}$
+	24	e^{-y}
−	0	$-e^{-y}$

    $$\Rightarrow -e^{-y}\left(y^4 + 4y^3 + 12y^2 + 24y + 24\right)\Big|_0^\infty = 0 - 24$$

13. We said that we could use the Poisson distribution to approximate the Binomial distribution. Under what circumstances, and exactly how do we do it (given the parameters for the Binomial, what are the parameters for the Poisson)?

    Poisson Approx to Binomial is typically done when θ is small (or large, since the Binomial also works with $1 - \theta$) and n is large. In that case, we take $\lambda = n\theta$ (Notice that that makes the two means the same).

    We said that we could use the normal distribution to approximate the Binomial distribution. Under what circumstances, and exactly how do we do it (given the parameters for the Binomial, what are the parameters for the normal)?

    Normal Approx to Binomial is typically done when θ is close to $\frac{1}{2}$ (so it is "moderate", and the Binomial is close to symmetric), and n is large. In that case, $\mu = n\theta$ and $\sigma^2 = n\theta(1 - \theta)$.

14. Use the Maclaurin series for the mgf $M(t)$ in order to find the 4$^{\text{th}}$ moment about the mean, if

    $$M(t) = \frac{1}{1 - t^2}$$

    (Hint: Think Geo Series)

    Similar to Exercise 4.38:

    The series can be found by first taking the series for $1/(1 - t)$. Notice that this is the sum of a geometric series:

    $$\frac{1}{1 - t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n$$

substitute t^2 for t to get our series:

$$\frac{1}{1-t^2} = 1 + t^2 + t^4 + t^6 + \cdots$$

Recall that:

$$\text{The coefficient of } t^n = \frac{M^{(n)}(0)}{n!}$$

In general, to get the fourth moment about the origin, we expand and simplify:

$$E((X-\mu)^4) = E(X^4) - 4E(X^3)\mu + 6E(X^2)\mu^2 - 4E(X)\mu^3 + \mu^4$$

But in this case $\mu = 0$, and the odd moments are zero, so we're just computing $E(X^4)$, which is

$$\frac{M^{(4)}(0)}{4!} = 1 \qquad \Rightarrow \qquad M^{(4)}(0) = 4!$$

(from the coefficient of t^4)

15. If $Y = aX + b$, and $M(t)$ is the mgf of X, find the mgf of Y:

    Recall Exercise 4.39.

    In this case,

    $$M_Y(t) = E(\mathrm{e}^{tY}) = E(\mathrm{e}^{t(aX+b)}) = E(\mathrm{e}^{X(at)}\mathrm{e}^{bt})$$

    And, e^{bt} is constant, so it can be factored out of the expected value:

    $$M_Y(t) = \mathrm{e}^{bt}E(\mathrm{e}^{X(at)}) = \mathrm{e}^{bt}M_X(at)$$

16. Recall the series expansion of the exponential function, and use it to compute the expected value of $X(X-1)$.

    Typo: Question should be about the Poisson Distribution

    With Poisson, we compute $E(X(X-1))$ (We did this computation in Exercise 5.33):

    $$E(X(X-1)) = \mathrm{e}^{-\lambda}\sum_{x=0}^{\infty}\frac{x(x-1)\lambda^x}{x!} = \mathrm{e}^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^2\cdot\lambda^{x-2}}{(x-2)!} =$$

    $$\mathrm{e}^{-\lambda}\lambda^2\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} = \mathrm{e}^{-\lambda}\lambda^2\mathrm{e}^{\lambda} = \lambda$$

17. Fill in the question marks: If the mgf is:

    $$M(t) = \mathrm{e}^{3t+8t^2}$$

    then the random variable was a normal distribution with mean 2 and standard deviation: $\frac{1}{2}\sigma^2 = 8$, so $\sigma = 4$

Given your answer, show what you get for the mgf of Z (and check the result of the previous exercise) if you let $Z = \frac{X-\mu}{\sigma}$

NOTE: Take a look at Homework problem 4.40, this is similar- In fact, the answer is the standard normal, so you should get:

$$e^{\frac{1}{2}t^2}$$

18. Let X be a normally distributed random variable with $\mu = 124$ and $\sigma = 7.5$.

    Use Chebyshev's Inequality to estimate the probability that $64 < X < 184$.

    Use Table III to get the probability that $64 < X < 184$.

    First compute k:

    $$X - \mu = k\sigma \quad \Rightarrow \quad k = \frac{60}{7.5} = 8$$

    Therefore,

    $$P(64 < X < 184) = P(|X - 124| \le 8 \cdot 7.5) \ge 1 - \frac{1}{8^2} = 0.984375$$

    Using the actual normal distribution with $Z = \frac{X-124}{7.5}$,

    $$P(64 < X < 184) = P(|Z| < 8) \approx 1$$

    (Whoops! I should have used a smaller k)

19. A company employs n people. Its income is a Gamma distribution, $g(x; 80\sqrt{n}, 2)$ Its cost is $8n$. Find the number of people that will give the maximum expected profit (Income−Cost).

    This is Problem 6.52. The expected value of the gamma distribution is $\alpha\beta$, which in this case is $160\sqrt{n}$. Subtracting sales cost,

    $$E = 160\sqrt{n} - 8n \qquad \Rightarrow \quad E' = \frac{80}{\sqrt{n}} - 8 = 0 \quad \Rightarrow \quad n = 100$$

    Notice that E'' is negative for all $n > 0$, so E is concave down for $n > 0$, and the critical point represents a global maximum.

20. If X is a normal rv, $n(x; 2, 3)$, then use Table III to give:

    - $P(X \ge 4)$

      $$X \ge 4 \quad \Rightarrow \quad \frac{X-2}{3} \ge \frac{2}{3}$$

      so the probability is found by

      $$P(Z \ge 2/3) = \frac{1}{2} - P(0 < Z < 2/3) = 0.5 - 0.2486 = 0.2514$$

- $P(X \geq 1)$

$$X \geq 1 \quad \Rightarrow \quad \frac{X-2}{3} \geq \frac{-1}{3}$$

Therefore,

$$P\left(Z \geq \frac{-1}{3}\right) = \frac{1}{2} + P(0 < Z < 1/3) = 0.5 + 0.1293 = 0.6293$$

- $P(-1 < X < 4)$

$$-1 < X < 4 \quad \Rightarrow \quad -1 < \frac{X-2}{3} < \frac{2}{3}$$

This is

$$P(0 < Z < 1) + P(0 < Z < 2/3) = 0.3413 + 0.2486 = 0.5899$$

21. Compute the covariance of $f(x,y) = \frac{1}{4}(2x+y)$, $0 < x < 1$, $0 < y < 2$ (zero elsewhere).

See Exercise 4.44:

We might compute the marginal distributions first:

$$f_1(x) = \frac{1}{4}\int_0^2 2x + y\,dy = \frac{1}{4}\left(2xy + \frac{1}{2}y^2\right|_0^2 = \frac{1}{4}(4x + 2 - 0) = x + \frac{1}{2}$$

$$f_2(y) = \frac{1}{4}\int_0^1 2x + y\,dx = \frac{1}{4}\left(x^2 + xy\right|_0^1 = \frac{1}{4}(y + 1 - 0) = \frac{1}{4}(y+1)$$

Now, respective means:

$$\mu_x = \int_0^1 x(x+1/2)\,dx = \int_0^1 x^2 + \frac{1}{2}x\,dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$\mu_y = \frac{1}{4}\int_0^2 y(y+1)\,dy = \frac{1}{4}\left(\frac{1}{3}y^3 + \frac{1}{2}y^2\right|_0^2 = \frac{7}{6}$$

We also need $E(XY)$:

$$E(XY) = \frac{1}{4}\int_0^1\int_0^2 xy(2x+y)\,dy\,dx = \int_0^1 \frac{2}{3}x + x^2\,dx = \frac{2}{3}$$

Therefore,

$$\sigma_{xy} = E(XY) - \mu_x\mu_y = \frac{2}{3} - \frac{7}{12}\cdot\frac{7}{6} = -\frac{1}{72}$$

22. Given a standard normal distribution, find the Maclaurin series for its moment generating function. Use it to find formulas for the third, fourth and sixth moments about the origin.

Also see Exercise 6.38:

Recall that the Maclaurin expansion of the exponential is:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

So, for the MGF of the standard normal distribution,

$$e^{\frac{1}{2}t^2} = 1 + \frac{\left(\frac{1}{2}t^2\right)}{1} + \frac{\left(\frac{1}{2}t^2\right)^2}{2} + \frac{\left(\frac{1}{2}t^2\right)^3}{3!} + \cdots$$

The general term of which is:

$$\frac{\left(\frac{1}{2}t^2\right)^k}{k!} = \frac{1}{2^k k!}t^{2k}$$

Recall that we said:

$$\text{Coefficient of } t^n = \frac{M^{(n)}(0)}{n!}$$

Doing a comparison, we see that all the odd moments are zero. For the even moments, let $n = 2k$ or $k = n/2$, and we see that

$$M^{(n)}(0) = \frac{n!}{2^{n/2}\left(\frac{n}{2}\right)!}$$

Therefore, the third moment is zero,

$$E(X^4) = \frac{4!}{2^2 2!} = 3$$

(Side remark: do you remember the number 3 in the coefficient of kurtosis? That's where it came from) and,

$$E(X^6) = \frac{6!}{2^3 3!} = \frac{4 \cdot 5 \cdot 6}{8} = 15$$

TYPO ON PROBLEMS 23 and 24! The memoryless property should be: $P(X \geq t + T | X \geq T) = P(X \geq t)$

23. Show that the exponential distribution is memoryless. That is, that

$$P(X \geq t + T | X \geq T) = P(X \geq T)$$

This is Exercise 6.16:

First, the CDF is:

$$G(T,\theta) = \frac{1}{\theta}\int_0^T \mathrm{e}^{-t/\theta}\,dt = 1 - \mathrm{e}^{-T/\theta}$$

so

$$P(X \geq t) = 1 - G(t,\theta) = \mathrm{e}^{-t/\theta} \qquad P(X \geq t+T) = \mathrm{e}^{-t/\theta}\mathrm{e}^{-T/\theta}$$

The conditional probability may need an explanation- Here, the events are $X \geq t+T$ and $X \geq T$. We see that one event is contained within the other:

$$P(X \geq t+T | X \geq T) = \frac{P(X \geq t+T)}{P(X \geq T)} = \frac{\mathrm{e}^{-t/\theta}\mathrm{e}^{-T/\theta}}{\mathrm{e}^{-T/\theta}} =$$

$$\mathrm{e}^{-t/\theta} = P(X \geq t)$$

24. Show that the geometric distribution is memoryless. That is,

$$P(X \geq t+T | X \geq T) = P(X \geq t)$$

(Typo in the original)

We will do a similar thing to Problem 23. That is,

$$P(X \geq t+T | X \geq T) = \frac{P(X \geq t+T)}{P(X \geq T)} = \frac{1 - P(X \leq t+T)}{1 - P(X \leq T)} = \frac{1 - F(t+T)}{1 - F(T)}$$

where $F(t)$ is the CDF of the geometric distribution, which we compute below:

$$F(t) = P(X \leq t) = \sum_{x=1}^{t} \theta(1-\theta)^{x-1} = \frac{\theta}{1-\theta}\sum_{x=1}^{t}(1-\theta)^x$$

Do we remember the partial sum of a geometric series?

$$(1-r)(r + r^2 + \ldots r^t) = r - r^{t+1} \qquad \Rightarrow \qquad \sum_{x=1}^{t} r^x = \frac{r - r^{t+1}}{1-r} = \frac{r(1-r^t)}{1-r}$$

Substituting $r = 1-\theta$, we get:

$$\frac{\theta}{1-\theta}\sum_{x=1}^{t}(1-\theta)^x = \frac{\theta}{1-\theta}\cdot\frac{(1-\theta)(1-(1-\theta)^t)}{1-(1-\theta)} = 1 - (1-\theta)^t$$

Now, $1 - F(t) = (1-\theta)^t$, and continuing where we left off:

$$P(X \geq t+T | X \geq T) = \frac{1 - F(t+T)}{1 - F(T)} = \frac{(1-\theta)^{t+T}}{(1-\theta)^T} = (1-\theta)^t = P(X \geq t)$$

25. Let X_1 be a uniform distribution on the interval $[1, 5]$, let X_2 be a normal distribution, $n(x; 3, 2)$ and let X_3 be another normal distribution, $n(x; -1, 3)$. Find the mean and variance of $Y = 2X_1 - 3x_2 + X_3$, assuming that X_1, X_2, X_3 are independent.

    Main idea here:

    $$\mu_Y = 2\mu_{X_1} - 3\mu_{X_2} + \mu_{X_3}$$

    And, since they are independent (covariances are all zero):

    $$\sigma_Y^2 = 4\sigma_{X_1}^2 + 9\sigma_{X_2}^2 + \sigma_{x_3}^2$$

    And, substitute the numbers in to get the numerical answers.

26. Find the moment generating function for the geometric distribution, and use it to find the first two moments about the origin.

    Exercise 5.20:

    $$M(t) = E(\mathrm{e}^{tx}) = \sum_{x=1}^{\infty} \mathrm{e}^{tx}\theta(1-\theta)^{x-1} = \frac{\theta}{1-\theta}\sum_{x=1}^{\infty}\left(\mathrm{e}^t(1-\theta)\right)^x$$

    Recall that the sum of the geometric series (pay attention to the starting index):

    $$\sum_{n=k}^{\infty} = \frac{r^k}{1-r}$$

    Therefore, with $r = \mathrm{e}^t(1-\theta)$, we have the moment generating function:

    $$M(t) = E(\mathrm{e}^{tx}) = \frac{\theta}{1-\theta} \cdot \frac{\mathrm{e}^t(1-\theta)}{1-\mathrm{e}^t(1-\theta)} = \frac{\theta\mathrm{e}^t}{1-\mathrm{e}^t(1-\theta)}$$

    Take the derivative with respect to t (a little messy), set $t = 0$ to get:

    $$M'(0) = \frac{1}{\theta} \qquad M''(0) = \frac{2-\theta}{\theta^2}$$