Final Exam: Operations Research

Instructions:

1. **Personalization:** At the top of your solutions, write down the **last digit of your Student ID**. We will call this number K.
   - *Example: If your ID ends in 7, then* $K = 7$.
   - Several questions will ask you to use K to modify the problem values.
2. The use of AI is expressly forbidden, and will constitute a violation of the College's policy on plagiarism.
3. You may use the book, your notes, and anything on our class website. No other materials are allowed.
4. You may use a calculator or spreadsheet, but no other computational aid, including online calculators or your own computer code.
5. Show all work using the notation and conventions used in this course. The use of notation or vocabulary that we did not use in class or in our book will be taken as evidence of using materials that were not allowed.
6. Unsupported answers receive little or no credit.
7. All solutions should be neatly written up (not typeset), step-by-step, specifically using notation and algorithms that we used in class.
8. **Due date:** Solutions will be due by 11:59 PM on Wednesday. I will not accept late work, so if there are other reasons you may not make that deadline, you need to get permission in advance.
9. Expectation: I expect it will take approximately 3 hours total, but that will depend on how much you need to look up. Try to solve a couple of problems at one sitting, but then take a few days to do them all.

Point breakdown:

Question	Points
1	5
2	15
3	20
4	10

Question	Points
5	20
6	15
7	15
	100 total

Short Answer

1. A student posts the following summary of the Fundamental Theorem of Linear Programming on the class forum:

   > *"The Fundamental Theorem states that if an LP has an optimal solution, there is always exactly one unique corner point that is the optimal solution."*

   (a) Identify the **two specific words** that make this statement mathematically false.

   (b) Draw a simple 2D sketch (axes and constraints) of a feasible region that proves the student's statement is wrong. Clearly label the optimal solution(s) on your sketch.

2. In this problem you are trying to reconstruct as much as possible about a **primal** linear program (a "max" problem) given only **partial** information about its dual.

   You are told that the dual problem has the following form:

   $$\min w = 10y_1 + 5y_2 + 3y_3$$

   where

   $$y_1 \text{ is unrestricted (urs)}, \qquad y_2 \geq 0, \qquad y_3 \leq 0$$

   No other information about the dual is given.

   (a) Suppose the primal has decision variables $x_1, \ldots, x_n$ and is written in the usual way. Based on the information about the dual given above, determine *everything you can say with certainty* about:

   - the **number** of constraints in the primal,
   - the **type** of each constraint (i.e., "$\leq$", "$\geq$", or "$=$"),
   - the **right-hand side** of each constraint.

   Clearly separate what is *determined* from what is *not determined* by the given dual.

   (b) Give a concrete example of a possible primal problem that is consistent with information about the dual given previously.

   (c) Starting from your primal in part (b), write the dual to verify that the conditions given are satisfied. Show enough work so that a reader could follow how each piece of the dual came from your primal.

   (d) In class we discussed that an equality constraint in the primal corresponds to an unrestricted dual variable. Explain this to a non-math major using the idea of *slack* or *surplus*. Your explanation should be 3–5 sentences in plain language. (Hint: You might think about what happens with a $\leq$ constraint first as a point of comparison.)

Linear Programming & Sensitivity

3. Consider the LP below and the *incorrect* final simplex tableau:

$$\begin{aligned} \text{Max } z = & 3x_1 + 4x_2 + x_3 \\ & 2x_1 + x_2 + x_3 \quad \leq 10, \\ & x_1 + 3x_2 + 2x_3 \quad \leq 15, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

x_1	x_2	x_3	RHS
0	0	3	25
0	1	1	5
1	0	-1	2

(a) Identify and explain all errors using the computations and vocabulary used in class.

(b) Suppose the true optimal basis has x_1 and x_2 basic. Compute the correct optimal solution, optimal value, allowable range for c_1, and shadow prices using our course conventions.

(c) Explain how the optimal value changes if the RHS of constraint 2 increases by 1.

Transportation

4. Consider the transportation tableau given below:

	D_1	D_2	D_3	D_4	Supply
S_1	4 20	6 20	8	8	40
S_2	6	8	6 10	7 50	60
S_3	5	7 10	6 40	8	50
Demand	20	30	50	50	

Scenario: Management is considering hiring a new trucking company for the route from S_1 to D_3. The new trucking company offers a rate of **$(8 - K)** per unit for this route. (Recall K is the last digit of your ID).

(a) Calculate the shadow price for cell $(1, 3)$ using the (u_i, v_j) from the optimal tableau.

(b) Based on your calculation, would this new price make the route profitable to use? (If YES: Which currently active route would likely see a decrease in shipping volume?, If NO: By how much more would they need to lower the price to break into the optimal solution?)

The Assignment Problem

5. A small gallery has four pieces of art it wishes to sell: $A_1,\ A_2,\ A_3,\ A_4$. Four collectors (C_1, C_2, C_3, C_4) have each submitted bids (in thousands of dollars) for each artwork. Each collector may purchase *at most one* piece, and each artwork may be sold to *at most one* collector.

   The bid matrix is given below.

	A_1	A_2	A_3	A_4
C_1	14	11	15	9
C_2	10	12	8	11
C_3	13	9	14	10
C_4	12	10	11	13

The gallery's goal is to assign artworks to collectors in a way that **maximizes total revenue**.

(a) Explain how this situation can be modeled as a linear program. Your answer should clearly specify: the decision variables, the objective function, and the assignment constraints.

(b) Then explain how a *maximization* assignment problem can be converted into an equivalent *minimization* problem suitable for the Hungarian Method, as implemented in this course, and give the appropriate assignment table.

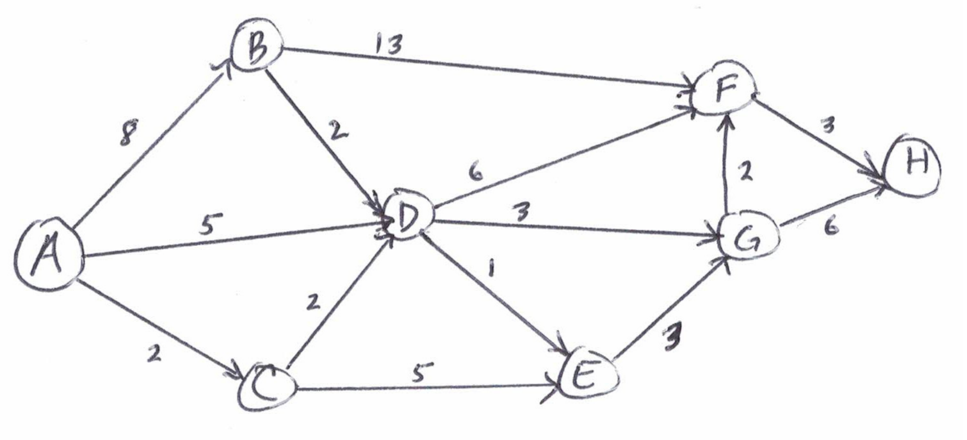
(c) Apply the Hungarian Method to the transformed problem. **Explicitly show** all of your steps in this process, including all row and column reductions, how you decide if it is optimal, and if it isn't, show any further matrix adjustments.

Finally, state the optimal assignment and the corresponding *maximum* total revenue for the gallery.

(d) Suppose collector C_3 increases their bid on artwork A_2 by \$4,000 (to \$13,000). Without fully re-solving the problem, discuss whether this increase *could* change the optimal assignment, and explain your reasoning. Your answer should focus on structure and reasoning rather than exhaustive computation.

Shortest Distance

6. Consider the following weighted, directed graph (starts at A, goes to H).



Modify the Graph: Recall that the K below refers to the last digit of your student ID.

- Change the weight of edge **A** → **C** from 2 to $2 + K$.
- Change the weight of edge **D** → **G** from 3 to $3 + K$.

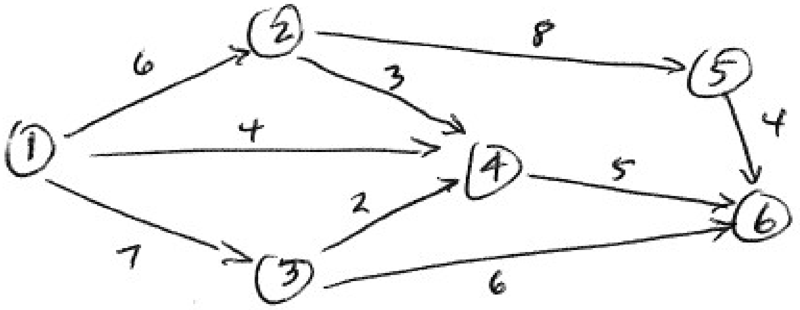
(a) Run Dijkstra's Algorithm using the technique we learned in class. Report the state of the algorithm at the exact moment when **Node G** gets boxed.

- Which nodes were *already* boxed before G?
- What are the distances of G's neighbors (E,D,F,H) after distances to G have been computed.

(b) Look at the edge connecting **Node F** → **Node H** (currently weight 3). Suppose this road is under construction and the travel time increases. **How high** would the weight of edge (F, H) have to rise before the *optimal* path from A to H switches to a different route? (Show your logic).

Maximum Flow

7. For this graph, you may assume that the source is node 1 and the terminal node is node 6.



Modify the Graph: Change the capacity of the edge **Node 3** → **Node 4** to be $2 + K$.

A junior analyst suggests that the minimum cut(A,B) uses $A = \{1, 2, 3\}$.

- Calculate the capacity of this specific cut using your personalized values.
- Does this cut capacity equal your Max Flow (and report your max flow)? If not, find a cut so that it is equal to the max flow (you might answer the next question first in that case).
- From your graph showing your (final) flow values, construct the residual graph using the rules we showed in class.