Example

$$\begin{array}{rll} \max & 6x_1 & +5x_2 \\ \text{s.t.} & x_1 & +x_2 \le 5 \\ & 3x_1 & +2x_2 \le 12 \end{array}$$

with $x_1, x_2 \geq 0$. Find an upper bound to the maximum.

- Using constraint 2 (multiply by 3):

$$6x_1 + 5x_2 \le 9x_1 + 6x_2 \le 36$$

- Using constraint 1 (multiply by 6):

$$6x_1 + 5x_2 \le 6x_1 + 6x_2 \le 30$$

Example

$$\begin{array}{rll} \max & 6x_1 & +5x_2 \\ \text{s.t.} & x_1 & +x_2 \leq 5 \\ & 3x_1 & +2x_2 \leq 12 \end{array}$$

How about a linear combination?

$$\begin{array}{rrl} y_1(x_1 & +x_2) & \leq 5y_1 \\ +y_2(3x_1 & +2x_2) & \leq 12y_2 \\ \hline (y_1+3y_2)x_1 & +(y_1+2y_2)x_2 & \leq 5y_1+12y_2 \end{array}$$

What needs to be true for this to give an upper bound? (The Dual)

$$\begin{array}{rl} \min & 5y_1 + 12y_2 \\ \text{s.t.} & y_1 + 3y_2 \geq 6 \\ & y_1 + 2y_2 \geq 5 \end{array}$$

with $y_1, y_2 \geq 0$.

We showed that the primal, dual are:

$$\begin{array}{rlll} \max & 6x_1 & +5x_2 \\ \text{s.t.} & x_1 & +x_2 \leq 5 \\ & 3x_1 & +2x_2 \leq 12 \end{array} \quad \Longleftrightarrow \quad \begin{array}{rl} \min & 5y_1 + 12y_2 \\ \text{s.t.} & y_1 + 3y_2 \geq 6 \\ & y_1 + 2y_2 \geq 5 \end{array}$$

Generalized, we have a "normal" max problem:

$$\begin{array}{rl} \max z = & \mathbf{c}^T\mathbf{x} \\ \text{st} & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{rl} \min w = & \mathbf{b}^T\mathbf{y} \\ \text{st} & A^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{array}$$

Note: **b** may have negative values in this construction.

$$\begin{aligned} \max z = \quad & \mathbf{c}^T\mathbf{x} \\ \text{st} \quad & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{aligned} \quad \iff \quad \begin{aligned} \min w = \quad & \mathbf{b}^T\mathbf{y} \\ \text{st} \quad & A^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge 0 \end{aligned}$$

How many constraints are in the dual?

Same as number of variables in primal.

How many variables are in the dual?

Same as number of constraints in primal.

- Every primal has a dual.
- A maximum becomes a minimum (dual of dual is primal).
- Variable in primal $\rightarrow$ constraint in dual.
- Constraint in primal $\rightarrow$ variable in dual.
- The RHS of primal $\rightarrow$ obj function coeffs in dual.

Self test: Create the Dual

$$
\begin{array}{rllll}
\max & 2x_1 & +3x_2 & -x_3 & \\
\text{s.t.} & x_1 & +2x_2 & +x_3 & \le 1 \\
& x_1 & -x_2 & -x_3 & \le 5
\end{array}
\iff
\begin{array}{rl}
\min & y_1 + 5y_2 \\
\text{s.t.} & y_1 + y_2 \ge 2 \\
& 2y_1 - y_2 \ge 3 \\
& y_1 - y_2 \ge -1
\end{array}
$$

An example with non-"normal" issues. Find the dual for:

$$
\begin{aligned}
\min \quad & 8x_1 + 5x_2 + 4x_3 \\
\text{s.t.} \quad & 4x_1 + 2x_2 + 8x_3 = 12 \\
& 7x_1 + 5x_2 + 6x_3 \geq 9 \\
& 8x_1 + 5x_2 + 4x_3 \leq 10 \\
& 3x_1 + 7x_2 + 9x_3 \geq 7 \\
& x_1 \geq 0, x_2 \text{ URS}, x_3 \leq 0
\end{aligned}
$$

We'll put this system into "normal form":

- Change the min to a max.
- Multiply Constraints 2 and 4 by -1.
- Change $=$ into inequalities.
- Find substitutions for x_2, x_3.

First three issues:

$$
\begin{array}{lrrrl}
\max & -8x_1 & -5x_2 & -4x_3 & \\
\text{s.t.} & 4x_1 & +2x_2 & +8x_3 & \leq 12 \\
 & -4x_1 & -2x_2 & -8x_3 & \leq -12 \\
 & -7x_1 & -5x_2 & -6x_3 & \leq -9 \\
 & 8x_1 & +5x_2 & +4x_3 & \leq 10 \\
 & -3x_1 & -7x_2 & -9x_3 & \leq -7
\end{array}
$$

$$x_1 \geq 0, x_2 \text{ URS}, x_3 \leq 0$$

Last issue: Let $x_2 = x_4 - x_5$ and $x_3 = -x_6$ (with all these new vars non-neg)

Now a summary of the equations using a table:

$$
\begin{array}{rrrr|l}
x_1 \geq 0 & x_4 \geq 0 & x_5 \geq 0 & x_6 \geq 0 & \\
\hline
4 & 2 & -2 & -8 & \leq 12 \\
-4 & -2 & 2 & 8 & \leq -12 \\
-7 & -5 & 5 & 6 & \leq -9 \\
8 & 5 & -5 & -4 & \leq -12 \\
-3 & -7 & 7 & 9 & \leq -7 \\
\hline
\geq -8 & \geq -5 & \geq 5 & \geq 4 &
\end{array}
$$

This is standard form. This table is given in the form:

$$
\begin{array}{c|c}
\mathbf{x} & \\
\hline
A & \mathbf{b} \\
\hline
\mathbf{c} &
\end{array}
\quad \Rightarrow \quad
\begin{array}{rl}
\min w = & \mathbf{b}^T\mathbf{p} \\
\text{st} & A^T\mathbf{p} \geq \mathbf{c} \\
& \mathbf{p} \geq 0
\end{array}
$$

Note that $\mathbf{p}$ has 5 variables (from the 5 constraints).

Let $p_1, \ldots, p_5$ be the new vars; we have 4 constraints (we had 4 variables in the primal).

$$
\begin{aligned}
\min w = \quad & 12p_1 - 12p_2 - 9p_3 + 10p_4 - 7p_5 \\
& 4p_1 - 4p_2 - 7p_3 + 8p_4 - 3p_5 \geq -8 \\
& 2p_1 - 2p_2 - 5p_3 + 5p_4 - 7p_5 \geq -5 \\
& -2p_1 + 2p_2 + 5p_3 - 5p_4 + 7p_5 \geq 5 \\
& -8p_1 + 8p_2 + 6p_3 - 4p_4 + 9p_5 \geq 4 \\
& p_1, p_2, p_3, p_4, p_5 \geq 0
\end{aligned}
$$

In the next frame, we'll start simplifying this system. Look to see what we might do first.

- Variables p_1, p_2 can be combined: $p_6 = p_1 - p_2$ (now is URS).
- Constraints 2 and 3 can be combined into equality.
- Multiply first constraint by -1 to get positive b (next page).

$$\begin{aligned}
\min w = \quad & 12(p_1 - p_2) - 9p_3 + 10p_4 - 7p_5 \\
& 4(p_1 - p_2) - 7p_3 + 8p_4 - 3p_5 \geq -8 \\
& 2(p_1 - p_2) - 5p_3 + 5p_4 - 7p_5 \geq -5 \\
& -2(p_1 - p_2) + 5p_3 - 5p_4 + 7p_5 \geq 5 \\
& -8(p_1 - p_2) + 6p_3 - 4p_4 + 9p_5 \geq 4 \\
& p_1, p_2, p_3, p_4, p_5 \geq 0
\end{aligned}$$

$$\begin{array}{rlllll}
\min w = & 12p_6 & -9p_3 & +10p_4 & -7p_5 & \\
& -4p_6 & +7p_3 & -8p_4 & +3p_5 & \leq 8 \\
& -2p_6 & +5p_3 & -5p_4 & +7p_5 & = 5 \\
& -8p_6 & +6p_3 & -4p_4 & +9p_5 & \geq 4
\end{array}$$

with p_6 URS, and $p_3, p_4, p_5 \geq 0$.

- Let $p_7 = -p_6$ and $p_8 = -p_4$
- p_7 is URS, $p_8 \leq 0$, $p_3, p_5 \geq 0$.

Changing notation slightly:

$$\begin{aligned} \max z = \quad & 12x_1 + 9x_2 + 10x_3 + 7x_4 \\ \text{s.t.} \quad & 4x_1 + 7x_2 + 8x_3 + 3x_4 \leq 8 \\ & 2x_1 + 5x_2 + 5x_3 + 7x_4 = 5 \\ & 8x_1 + 6x_2 + 4x_3 + 9x_4 \geq 4 \\ & x_1 \text{ URS}, x_2 \geq 0, x_3 \leq 0, x_4 \geq 0 \end{aligned}$$

$$\begin{aligned} \min w = \quad & 8y_1 + 5y_2 + 4y_3 \\ \text{s.t.} \quad & 4y_1 + 2y_2 + 8y_3 = 12 \\ & 7y_1 + 5y_2 + 6y_3 \geq 9 \\ & 8y_1 + 5y_2 + 4y_3 \leq 10 \\ & 3y_1 + 7y_2 + 9y_3 \geq 7 \\ & y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0 \end{aligned}$$

Summary:

Primal:	Dual:	
max	min	normal
$\leq$ constraint	≥ 0 variable	normal
$\geq$ constraint	≤ 0 variable	
Equality constraint	URS variable	
≥ 0 variable	$\geq$ constraint	normal
≤ 0 variable	$\leq$ constraint	
URS variable	Equality constraint	

Primal-Dual Table

$$\begin{aligned} \max z = \ & \mathbf{c}^T\mathbf{x} \\ \text{st} \ & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min w = \ & \mathbf{b}^T\mathbf{y} \\ \text{st} \ & A^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge 0 \end{aligned}$$

Main concerns:
If the inequalities in the primal are not normal, what happens in the dual?
If the variables are not normal in the primal, what happens in the dual?
(and vice-versa)

$$\begin{array}{c|cccc|c}
 & x_1? & x_2? & \cdots & x_n? & \\
\hline
y_1? & & & & & b_1? \\
y_2? & & A & & & b_2? \\
\vdots & & & & & \vdots \\
y_m? & & & & & b_m? \\
\hline
 & c_1? & c_2? & \cdots & c_n? &
\end{array}$$

Using a table

Starting problem:

$$\begin{aligned} \min w = \quad & 8y_1 + 5y_2 + 4y_3 \\ \text{s.t.} \quad & 4y_1 + 2y_2 + 8y_3 = 12 \\ & 7y_1 + 5y_2 + 6y_3 \geq 9 \qquad \text{with } y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0 \\ & 8y_1 + 5y_2 + 4y_3 \leq 10 \\ & 3y_1 + 7y_2 + 9y_3 \geq 7 \end{aligned}$$

(Asterisks mark things that are different than "normal")

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	?8
y_2 urs$(*)$	2	5	5	7	?5
$y_3 \leq 0(*)$	8	6	4	9	?4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	?5
$y_3 \leq 0(*)$	8	6	4	9	?4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	?4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	$x_2 \geq 0$	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	$x_2 \geq 0$	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	$x_2 \geq 0$	$x_3 \leq 0$	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	$x_2 \geq 0$	$x_3 \leq 0$	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	$x_2 \geq 0$	$x_3 \leq 0$	$x_4 \geq 0$	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table- Here are Primal and Dual

	x_1 URS	$x_2 \geq 0$	$x_3 \leq 0$	$x_4 \geq 0$	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

$$\max z = \quad 12x_1 + 9x_2 + 10x_3 + 7x_4$$

$$\begin{aligned} \text{s.t.} \quad & 4x_1 + 7x_2 + 8x_3 + 3x_4 \leq 8 \\ & 2x_1 + 5x_2 + 5x_3 + 7x_4 = 5 \\ & 8x_1 + 6x_2 + 4x_3 + 9x_4 \geq 4 \end{aligned}$$

$$x_1 \text{ URS}, x_2 \geq 0, x_3 \leq 0, x_4 \geq 0$$

$$\min w = \quad 8y_1 + 5y_2 + 4y_3$$

$$\begin{aligned} \text{s.t.} \quad & 4y_1 + 2y_2 + 8y_3 = 12 \\ & 7y_1 + 5y_2 + 6y_3 \geq 9 \\ & 8y_1 + 5y_2 + 4y_3 \leq 10 \\ & 3y_1 + 7y_2 + 9y_3 \geq 7 \end{aligned}$$

$$y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0$$

Example 2

Use a table to write the dual:

$$\begin{aligned} \max z = \quad & 2x_1 + x_2 \\ \text{st} \quad & x_1 + x_2 = 2 \\ & 2x_1 - x_2 \geq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned}$$

	$x_1 \geq 0$	x_2 urs$(*)$	
y_1 ?	1	1	$= 2(*)$
y_2 ?	2	-1	$\geq 3(*)$
y_3 ?	1	-1	≤ 1
	? 2	? 1	

	$x_1 \geq 0$	x_2 urs$(*)$	
y_1 urs	1	1	$= 2(*)$
$y_2 \leq 0$	2	-1	$\geq 3(*)$
$y_3 \geq 0$	1	-1	≤ 1
	≥ 2	$= 1$	

Use a table to write the dual:

$$\begin{aligned} \max z = \ & 2x_1 + x_2 \\ \text{st} \ & x_1 + x_2 = 2 \\ & 2x_1 - x_2 \geq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned}$$

	$x_1 \geq 0$	x_2 urs$(*)$	
y_1 urs	1	1	$= 2(*)$
$y_2 \leq 0$	2	-1	$\geq 3(*)$
$y_3 \geq 0$	1	-1	≤ 1
	≥ 2	$= 1$	

$$\begin{aligned} \min w = \ & 2y_1 + 3y_2 + y_3 \\ \text{st} \ & y_1 + 2y_2 + y_3 \geq 2 \\ & y_1 - y_2 - y_3 = 1 \\ & y_1 \text{ urs}, y_2 \leq 0, y_3 \geq 0 \end{aligned}$$

Last Example

Find the dual associated with the following primal:

$$\begin{aligned} \max \quad & z = 3x_1 + x_2 \\ \text{st} \quad & 2x_1 + x_2 \leq 4 \\ & 3x_1 + 2x_2 \geq 6 \\ & 4x_1 + 2x_2 = 7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution:

	$x_1 \geq 0$	$x_2 \geq 0$	
$y_1 \geq 0$	2	1	≤ 4
$y_2 \leq 0$	3	2	$\geq 6(*)$
y_3 URS	4	2	$= 7(*)$
	≥ 3	≥ 1	

$$
\begin{array}{rl}
\max & z = 4y_1 + 6y_2 + 7y_3 \\
\text{st} & 2y_1 + 3y_2 + 4y_3 \geq 3 \\
& y_1 + 2y_2 + 2y_3 \geq 1 \\
& y_1 \geq 0, y_2 \leq 0, y_3 \text{ URS}
\end{array}
$$