The Simplex Method

- Standard form (max):

$$
\begin{aligned}
z - \mathrm{c}^T\mathrm{x} &= 0 \\
A\mathrm{x} &= \mathrm{b} \\
\mathrm{x} \geq 0, &\quad \mathrm{b} \geq 0
\end{aligned}
$$

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- Build initial tableau.

$$\begin{array}{c|c|c} 1 & -\mathsf{c}^T & 0 \\ \hline 0 & A & \mathsf{b} \end{array}$$

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$$\begin{aligned} z - c^T x &= 0 \\ Ax &= b \\ x \geq 0, &\quad b \geq 0 \end{aligned}$$

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$$\begin{array}{c|c|c} 1 & -c^T & 0 \\ \hline 0 & A & b \end{array}$$

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- Is the BFS optimal?

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- Is the BFS optimal?
  1. Yes- We're done.
  2. No- Find a (better) adjacent BFS.
- Repeat.

Example from Wednesday

LP in standard form:

$$\begin{array}{llllll} z & -6x_1 - & 5x_2 - & 0s_1 - & 0s_2 & = 0 \\ & x_1 + & x_2 + & s_1 & & = 5 \\ & 3x_1 + & 2x_2 + & & s_2 & = 12 \end{array}$$

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$$
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 & x_1+ & x_2+ & s_1 & & = 5 \\
 & 3x_1+ & 2x_2+ & & s_2 & = 12
\end{array}
$$

Convert to the tableau (left-most column is optional)

z	x_1	x_2	s_1	s_2	rhs
1	-6	-5	0	0	0
0	1	1	1	0	5
0	3	2	0	1	12

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Step 1: Initial BFS-

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Step 1: Initial BFS- If we have all of the columns of the identity matrix, those variables are set to BV, all others to NBV.

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0	3	2	0	1	12

Step 1: Initial BFS- If we have all of the columns of the identity matrix, those variables are set to BV, all others to NBV.
Initial BFS

$$x_1 = 0, x_2 = 0, s_1 = 5, s_2 = 12 \qquad z = 0$$

Continuing

Which variable should come in to give a better z?

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$$\Rightarrow \begin{aligned} s_1 &= 5 - x_1 \\ s_2 &= 12 - 3x_1 \end{aligned}$$

Continuing

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0	3	2	0	1	12

$$\Rightarrow \begin{aligned} s_1 &= 5 - x_1 & \quad x_1 &\le 5 \\ s_2 &= 12 - 3x_1 & \quad x_1 &\le 12/3 = 4 \end{aligned}$$

We can make x_1 as large as 4 (larger makes s_1 negative).

Continuing

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$$\Rightarrow \quad \begin{array}{lll} s_1 & = 5 - x_1 & x_1 \leq 5 \\ s_2 & = 12 - 3x_1 & x_1 \leq 12/3 = 4 \end{array}$$

We can make x_1 as large as 4 (larger makes s_1 negative).That means s_2 is set to zero (and becomes the NBV). Pivot in the first column, second row

After pivoting (note that Row 0 is also computed)

z	x_1	x_2	s_1	s_2	rhs
1	0	-1	0	2	24
0	0	$1/3$	1	$-1/3$	1
0	1	$2/3$	0	$1/3$	4

Current BFS: $x_1 = 4$, $x_2 = 0$, $s_1 = 1$, $s_2 = 0$. Optimal?

After pivoting (note that Row 0 is also computed)

z	x_1	x_2	s_1	s_2	rhs
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Bring x_2 in.

After pivoting (note that Row 0 is also computed)

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Bring x_2 in. From our list of BVs:

After pivoting (note that Row 0 is also computed)

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Current BFS: $x_1 = 4$, $x_2 = 0$, $s_1 = 1$, $s_2 = 0$. Optimal? No. Bring x_2 in. From our list of BVs:

$$x_1 = 4 - \frac{2}{3}x_2 \quad \Rightarrow \quad x_2 \leq \frac{4}{2/3} = 6$$

$$s_1 = 1 - \frac{1}{3}x_2 \quad \Rightarrow \quad x_2 \leq \frac{1}{1/3} = 3$$

Note where these values come from: "RHS/Col Entry". Choose the Row with the smaller value, and that gives the pivot row.

Pivot

z	x_1	x_2	s_1	s_2	rhs
1	0	0	3	1	27
0	0	1	3	-1	3
0	1	0	-2	1	2

This is the optimal tableau. The optimal solution is $x_1 = 2, x_2 = 3$ with $z = 27$.

The Simplex Method

1. Build initial tableau.

1	$-c^T$	0
0	A	b

2. Initial BFS from cols of identity

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    3.1 Choose the **column** most negative coef.

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   3.1 Choose the **column** most negative coef.
   3.2 Perform a "ratio test" by taking "RHS/Lead Coeff".

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    Exceptions: Ignore zeros and neg coeffs.
    Choose the **row** with the **smallest** ratio.

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1. Build initial tableau.

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3. Look at Row 0 for neg coeffs:
   3.1 Choose the **column** most negative coef.
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   3.3 Pivot using the column/row we found.

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3. Look at Row 0 for neg coeffs:
    3.1 Choose the **column** most negative coef.
    3.2 Perform a "ratio test" by taking "RHS/Lead Coeff". *Exceptions:* Ignore zeros and neg coeffs. Choose the **row** with the **smallest** ratio.
    3.3 Pivot using the column/row we found.
4. If there are no more negative coefficients in Row 0, we're done. (Other stopping criteria later)

Example 2

$$\begin{array}{rll} \min & 2x_1 + x_2 - 4x_3 & \\ \text{st} & 3x_1 - x_2 + 2x_3 & \leq 25 \\ & -x_1 - x_2 + 2x_3 & \leq 20 \\ & -x_1 - x_2 + x_3 & \leq 5 \end{array}$$

with all variables non-negative.

- Change the min to a max:

Example 2

$$\begin{array}{rlr} \min & 2x_1 + x_2 - 4x_3 & \\ \text{st} & 3x_1 - x_2 + 2x_3 & \leq 25 \\ & -x_1 - x_2 + 2x_3 & \leq 20 \\ & -x_1 - x_2 + x_3 & \leq 5 \end{array}$$

with all variables non-negative.

- Change the min to a max: $\max z = -2x_1 - x_2 + 4x_3$
- Now construct the tableau and proceed as usual.
  Be sure to change back to a min at the end.

$$\begin{array}{lrl} \max & z = -2x_1 - x_2 + 4x_3 & \\ \text{st} & 3x_1 - x_2 + 2x_3 & \leq 25 \\ & -x_1 - x_2 + 2x_3 & \leq 20 \\ & -x_1 - x_2 + x_3 & \leq 5 \end{array}$$

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs
1	2	1	−4	0	0	0	0
0	3	−1	2	1	0	0	25
0	−1	−1	2	0	1	0	20
0	−1	−1	1	0	0	1	5

Initial tableau is using s_1, s_2, s_3 as BV. From Row 0, next is x_3.
Perform the ratio test to find the pivot row:

$$\begin{array}{lrl} \max & z = -2x_1 - x_2 + 4x_3 & \\ \text{st} & 3x_1 - x_2 + 2x_3 & \leq 25 \\ & -x_1 - x_2 + 2x_3 & \leq 20 \\ & -x_1 - x_2 + x_3 & \leq 5 \end{array}$$

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs
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Initial tableau is using s_1, s_2, s_3 as BV. From Row 0, next is x_3. Perform the ratio test to find the pivot row:

$$25/2, \quad 20/2, \quad 5/1 \quad \Rightarrow \quad \text{Row } 3$$

−2	−3	0	0	0	4	20
5	1	0	1	0	−2	15
1	1	0	0	1	−2	10
−1	−1	1	0	0	1	5

-2	-3	0	0	0	4	20
5	1	0	1	0	-2	15
1	1	0	0	1	-2	10
-1	-1	1	0	0	1	5

Now bring in x_2,

-2	-3	0	0	0	4	20
5	1	0	1	0	-2	15
1	1	0	0	1	-2	10
-1	-1	1	0	0	1	5

Now bring in x_2, and perform the ratio test to find pivot row.

$$
\begin{array}{rrrrrr|r}
-2 & -3 & 0 & 0 & 0 & 4 & 20 \\
\hline
5 & 1 & 0 & 1 & 0 & -2 & 15 \\
1 & 1 & 0 & 0 & 1 & -2 & 10 \\
-1 & -1 & 1 & 0 & 0 & 1 & 5
\end{array}
$$

Now bring in x_2, and perform the ratio test to find pivot row.

$$15/1, 10/1$$

Why do we ignore the third row?

-2	-3	0	0	0	4	20
5	1	0	1	0	-2	15
1	1	0	0	1	-2	10
-1	-1	1	0	0	1	5

Now bring in x_2, and perform the ratio test to find pivot row.

$$15/1, 10/1$$

Why do we ignore the third row?
It says $x_3 = 5 + x_2$,

$$\begin{array}{rrrrrr|r} -2 & -3 & 0 & 0 & 0 & 4 & 20 \\ \hline 5 & 1 & 0 & 1 & 0 & -2 & 15 \\ 1 & 1 & 0 & 0 & 1 & -2 & 10 \\ -1 & -1 & 1 & 0 & 0 & 1 & 5 \end{array}$$

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No restriction on how large x_2 can be.

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Now bring in x_2, and perform the ratio test to find pivot row.

$$15/1, 10/1$$

Why do we ignore the third row?
It says $x_3 = 5 + x_2$,
No restriction on how large x_2 can be.

Summary: New pivot is $(2, 2)$ position.

After Row Reduction

1	0	0	0	3	-2	50
4	0	0	1	-1	0	5
1	1	0	0	1	-2	10
0	0	1	0	1	-1	15

Bring in s_3.

After Row Reduction

1	0	0	0	3	−2	50
4	0	0	1	−1	0	5
1	1	0	0	1	−2	10
0	0	1	0	1	−1	15

Bring in s_3. Ratio test?

After Row Reduction

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 3 & -2 & 50 \\ \hline 4 & 0 & 0 & 1 & -1 & 0 & 5 \\ 1 & 1 & 0 & 0 & 1 & -2 & 10 \\ 0 & 0 & 1 & 0 & 1 & -1 & 15 \end{array}$$

Bring in s_3. Ratio test?

$$s_1 = 5 + 0s_3$$

After Row Reduction

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 3 & -2 & 50 \\ \hline 4 & 0 & 0 & 1 & -1 & 0 & 5 \\ 1 & 1 & 0 & 0 & 1 & -2 & 10 \\ 0 & 0 & 1 & 0 & 1 & -1 & 15 \end{array}$$

Bring in s_3. Ratio test?

$$\begin{aligned} s_1 &= 5 + 0s_3 \\ x_2 &= 10 + 2s_3 \end{aligned}$$

After Row Reduction

1	0	0	0	3	−2	50
4	0	0	1	−1	0	5
1	1	0	0	1	−2	10
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Bring in s_3. Ratio test?

$$\begin{aligned} s_1 &= 5 + 0s_3 \\ x_2 &= 10 + 2s_3 \\ x_3 &= 15 + s_3 \end{aligned}$$

After Row Reduction

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 3 & -2 & 50 \\ \hline 4 & 0 & 0 & 1 & -1 & 0 & 5 \\ 1 & 1 & 0 & 0 & 1 & -2 & 10 \\ 0 & 0 & 1 & 0 & 1 & -1 & 15 \end{array}\right]$$

Bring in s_3. Ratio test?

$$\begin{aligned} s_1 &= 5 + 0s_3 \\ x_2 &= 10 + 2s_3 \\ x_3 &= 15 + s_3 \\ z &= 50 + 2s_3 \end{aligned}$$

Conclusion?

After Row Reduction

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 3 & -2 & 50 \\ \hline 4 & 0 & 0 & 1 & -1 & 0 & 5 \\ 1 & 1 & 0 & 0 & 1 & -2 & 10 \\ 0 & 0 & 1 & 0 & 1 & -1 & 15 \end{array}$$

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Conclusion? The LP is unbounded

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Conclusion? The LP is unbounded

s_3 can be increased without bound, AND that causes z to be unbounded.

After Row Reduction

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Bring in s_3. Ratio test?

$$\begin{aligned} s_1 &= 5 + 0s_3 \\ x_2 &= 10 + 2s_3 \\ x_3 &= 15 + s_3 \\ z &= 50 + 2s_3 \end{aligned}$$

Conclusion? The LP is unbounded

s_3 can be increased without bound, AND that causes z to be unbounded.

Direction of unboundedness?

$$x = \begin{bmatrix} 0 \\ 10 \\ 15 \\ 5 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Direction of Unboundedness

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 3 & -2 & 50 \\ \hline 4 & 0 & 0 & 1 & -1 & 0 & 5 \\ 1 & 1 & 0 & 0 & 1 & -2 & 10 \\ 0 & 0 & 1 & 0 & 1 & -1 & 15 \end{array} \rightarrow \quad s_1 = 5 + 0s_3$$

Direction of Unboundedness

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Direction of Unboundedness

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Direction of unboundedness?

$$x_1 = 0$$

Direction of Unboundedness

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 3 & -2 & 50 \\ \hline 4 & 0 & 0 & 1 & -1 & 0 & 5 \\ 1 & 1 & 0 & 0 & 1 & -2 & 10 \\ 0 & 0 & 1 & 0 & 1 & -1 & 15 \end{array} \rightarrow \begin{aligned} s_1 &= 5 + 0s_3 \\ x_2 &= 10 + 2s_3 \\ x_3 &= 15 + s_3 \\ z &= 50 + 2s_3 \end{aligned}$$

Direction of unboundedness?

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 10 + 2s_3 \end{aligned}$$

Direction of Unboundedness

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 3 & -2 & 50 \\ \hline 4 & 0 & 0 & 1 & -1 & 0 & 5 \\ 1 & 1 & 0 & 0 & 1 & -2 & 10 \\ 0 & 0 & 1 & 0 & 1 & -1 & 15 \end{array} \rightarrow \begin{aligned} s_1 &= 5 + 0s_3 \\ x_2 &= 10 + 2s_3 \\ x_3 &= 15 + s_3 \\ z &= 50 + 2s_3 \end{aligned}$$

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so $\mathrm{d} = (0, 2, 1, 0, 0, 1)^T$

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Let's look at other issues that might come up.

The Simplex Method

What other things can happen in the algorithm?

1. Initial BFS from cols of identity
2. Look at Row 0 for neg coeffs:
   2.1 Choose the **column** most negative coef.
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   Exceptions: Ignore zeros and neg coeffs.
   Choose the **row** with the **smallest** ratio.
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3. If there are no more negative coefficients in Row 0, we're done.

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\max \quad & 3x_1 - x_2 - 4x_3 \\
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Construct the tableau:

z	x_1	x_2	x_3	e_1	s_2	e_3	rhs
1	3	−1	−4	0	0	0	0
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"The Big-M Method" is later and will fix this.

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Example:

$$\begin{aligned} \max_x \quad & 6x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 3x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$

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Initial Tableau:

$$\begin{array}{rrrr|r} -6 & -4 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 5 \\ 3 & 2 & 0 & 1 & 12 \end{array}$$

Final(?) tableau: Solution is $x_1 = 4$, $x_2 = 0$, $s_1 = 1$ and $s_2 = 0$

$$\begin{array}{crcr|r} 0 & 0 & 0 & 2 & 24 \\ \hline 0 & 1/3 & 1 & -1/3 & 1 \\ 1 & 2/3 & 0 & 1/3 & 4 \end{array}$$

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Can we bring in x_2 as a basic variable?

We can bring in x_2 with no change to z:

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$$\Downarrow$$

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New solution: $x_1 = 2, x_2 = 3$ with $s_1 = s_2 = 0$

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Any other solutions?
(2D) Line between $(4, 0)$ and $(2, 3)$

Alternative Optimal Solutions

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- If two BFS are optimal, the line segment joining them is also optimal (by convexity).

Example

Consider the following “final” tableau:

z	x_1	x_2	x_3	x_4	rhs
1	0	0	0	2	2
0	1	0	-1	1	2
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Row 0 may have a 0 for x_3 (z doesn't change)
Entries in the column are all negative or zero, Ratio test fails.
How many solutions do we have?

$$\begin{aligned} x_1 &= 2 + x_3 \\ x_2 &= 3 + 2x_3 \\ x_3 &= x_3 \\ x_4 &= 0 \end{aligned}$$

Example

$$\begin{aligned} \min \quad & z = -x_1 + 2x_2 \\ \text{st} \quad & x_1 - x_2 \leq 1 \\ & x_1 - 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Proceed as usual:

$$\begin{array}{cccc|c} -1 & 2 & 0 & 0 & 0 \\ \hline 1 & -1 & 1 & 0 & 1 \\ 1 & -2 & 0 & 1 & 2 \end{array} \quad \Rightarrow \quad \begin{array}{cccc|c} 0 & 1 & 1 & 0 & 1 \\ \hline 1 & -1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 \end{array}$$

Interpretation?

There is an optimal solution:

$$(1, 0)$$

The feasible set is unbounded.

Two Types of Unboundedness

- The objective function is unbounded (as is the feasible region).
- The feasible region is unbounded, but the objective function is not.
  "The LP is unbounded if there is a negative coefficient in Row 0, and all the remaining elements in the column are negative or zero"