The Simplex Method

- Standard form (max):

$$\begin{aligned} z - c^T x &= 0 \\ Ax &= b \\ x \geq 0, \quad b &\geq 0 \end{aligned}$$

- Build initial tableau.

1	$-c^T$	0
0	A	b

- Find an initial BFS.
- Is the BFS optimal?
  1. Yes- We're done.
  2. No- Find a (better) adjacent BFS.
- Repeat.

Example from Wednesday

LP in standard form:

$$\begin{aligned} z \quad -6x_1 - \quad 5x_2 - \quad 0s_1 - \quad 0s_2 &= 0 \\ x_1 + \quad x_2 + \quad s_1 \quad &= 5 \\ 3x_1 + \quad 2x_2 + \quad s_2 &= 12 \end{aligned}$$

Convert to the tableau (left-most column is optional)

z	x_1	x_2	s_1	s_2	rhs
1	-6	-5	0	0	0
0	1	1	1	0	5
0	3	2	0	1	12

Step 1: Initial BFS- If we have all of the columns of the identity matrix, those variables are set to BV, all others to NBV.
Initial BFS

$$x_1 = 0, x_2 = 0, s_1 = 5, s_2 = 12 \qquad z = 0$$

Continuing

Which variable should come in to give a better z?
From Row 0, most negative var: x_1.
Should we replace s_1 or s_2 (we want to make x_1 as large as possible for the max)

z	x_1	x_2	s_1	s_2	rhs
1	-6	-5	0	0	0
0	1	1	1	0	5
0	3	2	0	1	12

$$\Rightarrow \quad \begin{aligned} s_1 &= 5 - x_1 & x_1 &\leq 5 \\ s_2 &= 12 - 3x_1 & x_1 &\leq 12/3 = 4 \end{aligned}$$

We can make x_1 as large as 4 (larger makes s_1 negative).That means s_2 is set to zero (and becomes the NBV). Pivot in the first column, second row

After pivoting (note that Row 0 is also computed)

z	x_1	x_2	s_1	s_2	rhs
1	0	-1	0	2	24
0	0	1/3	1	$-1/3$	1
0	1	2/3	0	1/3	4

Current BFS: $x_1 = 4$, $x_2 = 0$, $s_1 = 1$, $s_2 = 0$. Optimal? No.
Bring x_2 in. From our list of BVs:

$$x_1 = 4 - \frac{2}{3}x_2 \quad \Rightarrow \quad x_2 \leq \frac{4}{2/3} = 6$$

$$s_1 = 1 - \frac{1}{3}x_2 \quad \Rightarrow \quad x_2 \leq \frac{1}{1/3} = 3$$

Note where these values come from: "RHS/Col Entry". Choose the Row with the smaller value, and that gives the pivot row.

Pivot

z	x_1	x_2	s_1	s_2	rhs
1	0	0	3	1	27
0	0	1	3	-1	3
0	1	0	-2	1	2

This is the optimal tableau. The optimal solution is $x_1 = 2, x_2 = 3$ with $z = 27$.

The Simplex Method

1. Build initial tableau.

1	$-c^T$	0
0	A	b

2. Initial BFS from cols of identity
3. Look at Row 0 for neg coeffs:
   1. Choose the **column** most negative coef.
   2. Perform a "ratio test" by taking "RHS/Lead Coeff". *Exceptions:* Ignore zeros and neg coeffs. Choose the **row** with the **smallest** ratio.
   3. Pivot using the column/row we found.
4. If there are no more negative coefficients in Row 0, we're done. (Other stopping criteria later)

Example 2

$$\begin{array}{rl} \min & 2x_1 + x_2 - 4x_3 \\ \text{st} & 3x_1 - x_2 + 2x_3 \le 25 \\ & -x_1 - x_2 + 2x_3 \le 20 \\ & -x_1 - x_2 + x_3 \le 5 \end{array}$$

with all variables non-negative.

- Change the min to a max: $\max z = -2x_1 - x_2 + 4x_3$
- Now construct the tableau and proceed as usual. Be sure to change back to a min at the end.

$$\begin{array}{rl} \max & z = -2x_1 - x_2 + 4x_3 \\ \text{st} & 3x_1 - x_2 + 2x_3 \le 25 \\ & -x_1 - x_2 + 2x_3 \le 20 \\ & -x_1 - x_2 + x_3 \le 5 \end{array}$$

z	x_1	x_2	x_3	s_1	s_2	s_3	*rhs*
1	2	1	-4	0	0	0	0
0	3	-1	2	1	0	0	25
0	-1	-1	2	0	1	0	20
0	-1	-1	1	0	0	1	5

Initial tableau is using s_1, s_2, s_3 as BV. From Row 0, next is x_3. Perform the ratio test to find the pivot row:

$$25/2, \quad 20/2, \quad 5/1 \quad \Rightarrow \quad \text{Row 3}$$

−2	−3	0	0	0	4	20
5	1	0	1	0	−2	15
1	1	0	0	1	−2	10
−1	−1	1	0	0	1	5

Now bring in x_2, and perform the ratio test to find pivot row.

$$15/1, 10/1$$

Why do we ignore the third row?
It says $x_3 = 5 + x_2$,
No restriction on how large x_2 can be.

Summary: New pivot is $(2, 2)$ position.

After Row Reduction

1	0	0	0	3	−2	50
4	0	0	1	−1	0	5
1	1	0	0	1	−2	10
0	0	1	0	1	−1	15

Bring in s_3. Ratio test?

$$\begin{aligned} s_1 &= 5 + 0s_3 \\ x_2 &= 10 + 2s_3 \\ x_3 &= 15 + s_3 \\ z &= 50 + 2s_3 \end{aligned}$$

Conclusion? The LP is unbounded
s_3 can be increased without bound, AND that causes z to be unbounded.
Direction of unboundedness?

$$\mathrm{x} = \begin{bmatrix} 0 \\ 10 \\ 15 \\ 5 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Direction of Unboundedness

1	0	0	0	3	−2	50
4	0	0	1	−1	0	5
1	1	0	0	1	−2	10
0	0	1	0	1	−1	15

$$\rightarrow \quad \begin{aligned} s_1 &= 5 + 0s_3 \\ x_2 &= 10 + 2s_3 \\ x_3 &= 15 + s_3 \\ z &= 50 + 2s_3 \end{aligned}$$

Direction of unboundedness?

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 10 + 2s_3 \\ x_3 &= 15 + s_3 \\ s_1 &= 5 \\ s_2 &= 0 \\ s_3 &= 0 + s_3 \end{aligned} \quad \rightarrow \mathrm{x} = \begin{bmatrix} 0 \\ 10 \\ 15 \\ 5 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so $\mathrm{d} = (0, 2, 1, 0, 0, 1)^T$ (the other vector is a BFS)

Detect an Unbounded LP

In one column (for one variable), we need:

- The entry in Row 0 is negative
  Why? That means making the variable > 0 will increase z.
- The other entries are all zero or negative, with at least one value not zero.
  Why? The ratio test fails, and this implies that this variable can be increased without bound (and the remaining solution remains feasible).

Let's look at other issues that might come up.

The Simplex Method

What other things can happen in the algorithm?

1. Initial BFS from cols of identity
2. Look at Row 0 for neg coeffs:
   - 2.1 Choose the **column** most negative coef.
   - 2.2 Perform a "ratio test" by taking "RHS/Lead Coeff".
     Exceptions: Ignore zeros and neg coeffs.
     Choose the **row** with the **smallest** ratio.
   - 2.3 Pivot using the column/row we found.
3. If there are no more negative coefficients in Row 0, we're done.

$$\begin{array}{rll} \max & 3x_1 - x_2 - 4x_3 & \\ \text{st} & 3x_1 - x_2 + 2x_3 & \geq 25 \\ & -x_1 - x_2 + 2x_3 & \leq 20 \\ & -x_1 - x_2 + x_3 & \geq 5 \end{array}$$

Construct the tableau:

z	x_1	x_2	x_3	e_1	s_2	e_3	*rhs*
1	3	−1	−4	0	0	0	0
0	3	−1	2	−1	0	0	25
0	−1	−1	2	0	1	0	20
0	−1	−1	1	0	0	−1	5

No initial BFS!
"The Big-M Method" is later and will fix this.

The Simplex Method

What other things can happen in the algorithm?

1. Initial BFS from cols of identity
2. Look at Row 0 for neg coeffs:
   - 2.1 Choose the **column** most negative coef.
   - 2.2 Perform a "ratio test" by taking "RHS/Lead Coeff".
     Exceptions: Ignore zeros and neg coeffs.
     Choose the **row** with the **smallest** ratio.
   - 2.3 Pivot using the column/row we found.
3. If there are no more negative coefficients in Row 0, we're done.

Example:

$$\begin{array}{rll} \max_x & 6x_1 + & 4x_2 \\ \text{s.t.} & x_1 + & x_2 \leq 5 \\ & 3x_1 + & 2x_2 \leq 12 \\ & x_1, & x_2 \geq 0 \end{array}$$

Initial Tableau:

−6	−4	0	0	0
1	1	1	0	5
3	2	0	1	12

Final(?) tableau: Solution is $x_1 = 4$, $x_2 = 0$, $s_1 = 1$ and $s_2 = 0$

0	0	0	2	24
0	1/3	1	−1/3	1
1	2/3	0	1/3	4

Can we bring in x_2 as a basic variable?

We can bring in x_2 with no change to z:

0	0	0	2	24
0	1/3	1	−1/3	1
1	2/3	0	1/3	4

$\Downarrow$

0	0	0	2	24
0	1	3	−1	3
1	0	−2	1	2

New solution: $x_1 = 2, x_2 = 3$ with $s_1 = s_2 = 0$
Any other solutions?
(2D) Line between $(4, 0)$ and $(2, 3)$

Alternative Optimal Solutions

- If a NBV in Row 0 is 0, and we can pivot in this column (and maintain the same value of z), then we may have alternative optimal solutions.
- If two BFS are optimal, the line segment joining them is also optimal (by convexity).

Example

Consider the following "final" tableau:

z	x_1	x_2	x_3	x_4	rhs
1	0	0	0	2	2
0	1	0	−1	1	2
0	0	1	−2	3	3

Interpretation?
Row 0 may have a 0 for x_3 (z doesn't change)
Entries in the column are all negative or zero, Ratio test fails.
How many solutions do we have?

$$\begin{aligned} x_1 &= 2 + x_3 \\ x_2 &= 3 + 2x_3 \\ x_3 &= x_3 \\ x_4 &= 0 \end{aligned}$$

Example

$$\begin{aligned} \min \quad & z = -x_1 + 2x_2 \\ \text{st} \quad & x_1 - x_2 \leq 1 \\ & x_1 - 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Proceed as usual:

−1	2	0	0	0
1	−1	1	0	1
1	−2	0	1	2

$\Rightarrow$

0	1	1	0	1
1	−1	1	0	1
0	−1	−1	1	1

Interpretation?

There is an optimal solution:

$$(1, 0)$$

The feasible set is unbounded.

Two Types of Unboundedness

- The objective function is unbounded (as is the feasible region).
- The feasible region is unbounded, but the objective function is not.
  "The LP is unbounded if there is a negative coefficient in Row 0, and all the remaining elements in the column are negative or zero"