Matrix Notation and Simplex (6.2)

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Consider the LP in standard form,

$$\begin{aligned} \max z = \quad & \mathbf{c}^T\mathbf{x} \\ \text{st} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

where A is $m \times n$ with rank m. We will re-partition these matrices and vectors.

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Let a subscript B denote the set of basic variables, and a subscript N denote the set of remaining variables.

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} \mathbf{c}_B \\ \mathbf{c}_N \end{bmatrix} \qquad A = [B \;\; N] \qquad \mathbf{b} = \mathbf{b}$$

Re-write the LP as:

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NOTE ABOUT TEXT: They take $\mathbf{c}$ as a *row* instead of the more usual column, so they do NOT use transposes.

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Given a set of basic variables for subscript B (and matrix B), then:

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The coefficients that from which we compute the max (or min) are in the vector $\mathbf{c}_B$. With A and Row 0 in the correct order:

$$\begin{array}{c|c} ?? & \mathbf{c}_B^T B^{-1}\mathbf{b} \\ \hline B^{-1}A & B^{-1}\mathbf{b} \end{array} \Rightarrow \begin{array}{c|c} -\mathbf{c}^T + \mathbf{c}_B^T B^{-1}A & \mathbf{c}_B^T B^{-1}\mathbf{b} \\ \hline B^{-1}A & B^{-1}\mathbf{b} \end{array}$$

We need to be careful that the variable indices line up for the remaining row zero coefficients (We'll see an example below).

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$$B\vec{e}_k = k^{\mathrm{th}} \text{ col of } B \doteq B_k$$

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- If x_j is an excess variable, then multiply previous by -1:

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- If x_j is an artificial variable, then $c_j = M$ and

$$\hat{c}_j = \mathbf{c}_B^T \left(B^{-1}\right)_k - M$$

Numerical Example 1:

"Dakota Furniture" (max):

x_1	x_2	x_3	s_1	s_2	s_3	rhs
-60	-30	-20	0	0	0	0
8	6	1	1	0	0	48
4	2	$\frac{3}{2}$	0	1	0	20
2	$\frac{3}{2}$	$\frac{1}{2}$	0	0	1	8

$\Rightarrow$

x_1	x_2	x_3	s_1	s_2	s_3	RHS
0	5	0	0	10	10	280
0	-2	0	1	2	-8	24
0	-2	1	0	2	-4	8
1	$\frac{5}{4}$	0	0	$-\frac{1}{2}$	$\frac{3}{2}$	2

If the set of basic variables (the basis) is s_1, x_3, x_1, then verify the previous computations.

SOLUTION: The matrix B is given by the columns of A corresponding to s_1, x_3, x_1.

$$B = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 3/2 & 4 \\ 0 & 1/2 & 2 \end{bmatrix} \qquad \Rightarrow \qquad B^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix}$$

Further, (note the order):

$$\mathbf{c}_B^T B^{-1}\mathbf{b} = 0 \cdot 24 + 20 \cdot 8 + 60 \cdot 2 = 280$$

The matrix B is made up of Columns 4, 3 and 1 of matrix A. Let N be made from columns 2, 5, 6 of matrix A (in that order). Then $A = [B, N]$ and

$$[0, 20, 60] \cdot B^{-1}[B, N] - [0, 20, 60, 30, 0, 0] = [0, 0, 0, 5, 10, 10]$$

Order was important in the last example. That is, if we use the original order, then

$$\mathbf{c}_B^T B^{-1} A - \mathbf{c}^T$$

would be meaningless-

$$[0, 20, 60]B^{-1}A - [60, 30, 20, 0, 0, 0] = [-58, -67.5, 0, 0, 39, -77]$$

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The coefficient is:

$$\mathbf{c}_B^T B^{-1} \mathbf{a}_2 - 30 = [0, 20, 60] \begin{bmatrix} -2 \\ -2 \\ 1.25 \end{bmatrix} - 30 = -40 + 75 - 30 = 5$$

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The coefficient is:

$$\mathbf{c}_B^T B^{-1}\vec{e}_2 - 0 = [0, 20, 60]\begin{bmatrix} 2 \\ 2 \\ -1/2 \end{bmatrix} - 30 = 40 - 30 - 0 = 10$$

Numerical Example 2:

This one is small enough to do by hand. Given that x_1, x_2 are the basic variables for the optimal tableau, find that tableau directly.

$$\begin{aligned} \max z = \quad & 3x_1 + x_2 \\ \text{st} \quad & 2x_1 - x_2 \le 2 \\ & -x_1 + x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

SOLUTION: Write the LP in standard form

x_1	x_2	s_1	s_2	rhs
-3	-1	0	0	0
2	-1	1	0	2
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So that the matrix B and B^{-1} are:

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So that the matrix B and B^{-1} are:

$$B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad B^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

- The optimal tableau is:

$$\begin{array}{c|c} \hline B^{-1}A & B^{-1}\mathbf{b} \end{array} = \begin{array}{cccc|c} \hline 1 & 0 & 1 & 1 & 6 \\ 0 & 1 & 1 & 2 & 10 \end{array}$$

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$$\begin{array}{cccc|c} 0 & 0 & 4 & 5 & 28 \\ \hline 1 & 0 & 1 & 1 & 6 \\ 0 & 1 & 1 & 2 & 10 \end{array}$$