6.5: Construction of the Dual

Here is the problem we'll work on:

$$\begin{aligned} \min_x \quad & 8x_1 + 5x_2 + 4x_3 \\ \text{s.t.} \quad & 4x_1 + 2x_2 + 8x_3 = 12 \\ & 7x_1 + 5x_2 + 6x_3 \ge 9 \\ & 8x_1 + 5x_2 + 4x_3 \le 10 \\ & 3x_1 + 7x_2 + 9x_3 \ge 7 \\ & x_1 \ge 0, x_2 \text{ URS}, x_3 \le 0 \end{aligned}$$

First we'll take care of the sign restrictions so that all of our variables are non-negative:

$$x_2 = x_4 - x_5, \qquad x_6 = -x_3, \qquad x_4, x_5, x_6 \ge 0$$

The LP now becomes:

$$\begin{aligned} \min_x \quad & 8x_1 + 5x_4 - 5x_5 - 4x_6 \\ \text{s.t.} \quad & 4x_1 + 2x_4 - 2x_5 - 8x_6 = 12 \\ & 7x_1 + 5x_4 - 5x_5 - 6x_6 \ge 9 \\ & 8x_1 + 5x_4 - 5x_5 - 4x_6 \le 10 \\ & 3x_1 + 7x_4 - 7x_5 - 9x_6 \ge 7 \\ & x_1, x_4, x_5, x_6 \ge 0 \end{aligned}$$

Now I'll multiply the objective function by -1 and change it to a maximum, and multiply constraints 2 and 4 by -1 (to get $\le$), and the equality is written as two constraints:

$$\begin{aligned} \max_x \quad & -8x_1 - 5x_4 + 5x_5 + 4x_6 \\ \text{s.t.} \quad & 4x_1 + 2x_4 - 2x_5 - 8x_6 \le 12 \\ & -4x_1 - 2x_4 + 2x_5 + 8x_6 \le -12 \\ & -7x_1 - 5x_4 + 5x_5 + 6x_6 \le -9 \\ & 8x_1 + 5x_4 - 5x_5 - 4x_6 \le 10 \\ & -3x_1 - 7x_4 + 7x_5 + 9x_6 \le -7 \\ & x_1, x_4, x_5, x_6 \ge 0 \end{aligned}$$

Now the dual as before, using the text's array. In the array to the left, we use p_i instead of y_i (we'll convert back later) for the "dual", and x_i for the "primal":

	$x_1 \ge 0$	$x_4 \ge 0$	$x_5 \ge 0$	$x_6 \ge 0$	
$p_1 \ge 0$	4	2	-2	-8	≤ 12
$p_2 \ge 0$	-4	-2	2	8	≤ -12
$p_3 \ge 0$	-7	-5	5	6	≤ -9
$p_4 \ge 0$	8	5	-5	-4	≤ -12
$p_5 \ge 0$	-3	-7	7	9	≤ -7
	≥ -8	≥ -5	≥ 5	≥ 4	

$\Rightarrow$

$$\begin{aligned} \min w = \quad & 12p_1 - 12p_2 - 9p_3 + 10p_4 - 7p_5 \\ & 4p_1 - 4p_2 - 7p_3 + 8p_4 - 3p_5 \ge -8 \\ & 2p_1 - 2p_2 - 5p_3 + 5p_4 - 7p_5 \ge -5 \\ & -2p_1 + 2p_2 + 5p_3 - 5p_4 + 7p_5 \ge 5 \\ & -8p_1 + 8p_2 + 6p_3 - 4p_4 + 9p_5 \ge 4 \\ & p_1, p_2, p_3, p_4 \ge 0 \end{aligned}$$

Change back to a maximum, then try to get all the cofficients in the problem positive. In particular, let $p_6 = p_2 - p_1$ so that p_6 is URS and $p_7 = -p_4$:

$$\max_p \quad 12p_6 + 9p_3 + 10p_7 + 7p_5$$

$$\begin{aligned}
&-4p_6 - 7p_3 - 8p_7 - 3p_5 \geq -8 \\
&-2p_6 - 5p_3 - 5p_7 - 7p_5 \geq -5 \\
&2p_6 + 5p_3 + 5p_7 + 7p_5 \geq 5 \\
&8p_6 + 6p_3 + 4p_7 + 9p_5 \geq 4 \\
&p_6 \text{ URS }, p_7 \leq 0, p_3, p_5 \geq 0
\end{aligned}$$

The two constraints ending with ± 5 on the right can be simplified to a single inequality. Let's re-write this expression and compare it to our original so that we can draw some conclusions about constructing the dual. As in the text, we will call the minimization problem the dual, so that we're constructing the primal (thus the change in the notation for the LPs):

$$\begin{aligned}
\max z = \quad & 12x_1 + 9x_2 + 10x_3 + 7x_4 \\
\text{s.t.} \quad & 4x_1 + 7x_2 + 8x_3 + 3x_4 \leq 8 \\
& 2x_1 + 5x_2 + 5x_3 + 7x_4 = 5 \\
& 8x_1 + 6x_2 + 4x_3 + 9x_4 \geq 4 \\
& x_1 \text{ URS}, x_3 \leq 0, x_2, x_4 \geq 0
\end{aligned}$$

$$\begin{aligned}
\min w = \quad & 8y_1 + 5y_2 + 4y_3 \\
\text{s.t.} \quad & 4y_1 + 2y_2 + 8y_3 = 12 \\
& 7y_1 + 5y_2 + 6y_3 \geq 9 \\
& 8y_1 + 5y_2 + 4y_3 \leq 10 \\
& 3y_1 + 7y_2 + 9y_3 \geq 7 \\
& y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0
\end{aligned}$$

Here is a list of things we should notice, going from the primal to the dual:

- The second constraint of the primal is =, the second variable in the dual y_2 is URS.
- The third constraint of the primal is "$\geq$", the third variable in the dual $y_3 \leq 0$.
- The first variable of the primal is URS, so the first constraint of the dual is =.
- The third variable of the primal is negative, the second constraint of the dual is "$\leq$" (not "normal").

And going from the dual to the primal:

- The first constraint of the dual is =, and the first variable of the primal is x_1 URS.
- The third constraint of the dual is $\leq$ (not normal), so the third variable of the primal is $x_3 \leq 0$.
- The second variable of the dual is URS, so the second constraint in the primal is =
- The third variable of the dual is $\leq$, so the third constraint of the primal is $\leq$.

In summary, we see that *the dual of the dual is the primal.*

The shortcut

Going from our original minimization problem (write the variables as y), we construct the following array with the information we know. Since we started with a minimization problem, we'll change the variables to y_i and fill in the bottom portion of the array. We'll put in question marks for the information we need to figure out, and (*) will denote what information is "not normal".

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	?8
y_2 urs$(*)$	2	5	5	7	?5
$y_3 \leq 0(*)$	8	6	4	9	?4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Look at the very bottom row. These will tell us how to categorize the variables at the top:

- If you see an equality constraint, the corresponding variable is URS (for the first column).
- Column 2 is "normal", so $x_2 \geq 0$.
- Column 3 is $\leq$, so $x_3 \leq 0$.
- Column 4 is again normal, so $x_4 \geq 0$.

Now look at the column on the right, and fill in the missing information on the left:

- In Row 1, y_1 is "normal", so on the right we have ≤ 8.
- In Row 2, y_2 is URS, so on the right we have $= 5$ (not normal).
- In Row 3, $y_3 \leq 0$ (not normal), so on the right we have ≤ 4.

This gives us the final array from which we can write the dual:

	$x_1$urs	$x_2 \geq 0$	$x_3 \leq 0$	$x_4 \geq 0$	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

$\Rightarrow$

$$\begin{aligned} \max z = \quad & 12x_1 + 9x_2 + 10x_3 + 7x_4 \\ \text{s.t.} \quad & 4x_1 + 7x_2 + 8x_3 + 3x_4 \leq 8 \\ & 2x_1 + 5x_2 + 5x_3 + 7x_4 = 5 \\ & 8x_1 + 6x_2 + 4x_3 + 9x_4 \geq 4 \\ & x_1 \text{ URS}, x_3 \leq 0, x_2, x_4 \geq 0 \end{aligned}$$

Summary of Conversions

	Primal (**max**)	Dual (**min**)
1	# constraints	# variables
2	# variables	# constraints
3	RHS	Obj Function
4	Obj Function	RHS
5	Coeff Matrix A	Coeff Matrix A^T
6	Eqn constraint	Unrestricted var
7	Unrestricted var	Eqn constraint
8	$\leq$ constraint	$\geq$ variable
9	$\geq$ constraint	$\leq$ variable
10	$\geq$ variable	$\geq$ constraint
11	$\leq$ variable	$\leq$ constraint