The Simplex Method

- Standard form (max) $z - \mathbf{c}^T\mathbf{x} = 0$ such that $A\mathbf{x} = \mathbf{b}$.
- Build initial tableau.

z	$-\mathbf{c}^T$	0
0	A	$\mathbf{b}$

- Find an initial BFS.
- Look at Row 0 for neg coeffs:
  1. Choose the **column** most negative coef.
  2. Perform a “ratio test” by taking “RHS/Lead Coeff”.
     Exceptions: Ignore zeros and neg coeffs.
     Choose the **row** with the **smallest** ratio.
  3. Pivot using the column/row we found.
  4. Repeat.

Stopping Criteria 1

Stop if all Row 0 coeffs are non-negative. The current solution is optimal and unique.
Example Final Tableau:

z	x_1	x_2	s_1	s_2	rhs
1	0	0	3	1	27
0	0	1	3	-1	3
0	1	0	-2	1	2

Maximizer: $x_1 = 2, x_2 = 3$ and $s_1 = s_2 = 0$. Notice also the top row:

$$z = 27 - 3s_1 - s_2$$

From Wed: Unbounded LP

Here was the final tableau for an unbounded region and an unbounded LP:

	z	x_1	x_2	x_3	s_1	s_2	s_3	rhs
	1	1	0	0	0	3	-2	50
s_1	0	4	0	0	1	-1	0	5
x_2	0	1	1	0	0	1	-2	10
x_3	0	0	0	1	0	1	-1	15

Translation: $\max z = 50 - x_1 - 3s_2 + 2s_3$ with

$$\mathbf{x} = \begin{bmatrix} 0 \\ 10 \\ 15 \\ 5 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example 3, A Little Foreshadowing

$$\begin{array}{rrrl} \max z = & x_1 & +x_2 & \\ \text{st} \quad & 2x_1 & +x_2 & \geq 4 \\ & x_1 & +2x_2 & = 6 \\ & x_1, & x_2 & \geq 0 \end{array}$$

Build the initial tableau.

z	x_1	x_2	e_1	rhs
1	-1	-1	0	0
0	2	1	-1	4
0	1	2	0	6

Not standard form... No initial BFS.

Choosing x_1, x_2 as the basic variables, we can perform the row reduction using an inverse matrix:

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1}[A|\mathbf{b}] = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\left[\begin{array}{ccr|c} 2 & 1 & -1 & 4 \\ 1 & 2 & 0 & 6 \end{array}\right] =$$

$$\left[\begin{array}{ccr|c} 1 & 0 & -2/3 & 2/3 \\ 0 & 1 & 1/3 & 8/3 \end{array}\right]$$

The new tableau:

$$\left[\begin{array}{c|ccr|r} z & x_1 & x_2 & e_1 & \text{rhs} \\ 1 & 0 & 0 & -1/3 & 10/3 \\ \hline 0 & 1 & 0 & -2/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & 8/3 \end{array}\right]$$

Bring e_1 into the set of BV (take x_2 out):

$$\left[\begin{array}{c|ccc|r} z & x_1 & x_2 & e_1 & \text{rhs} \\ 1 & 0 & 1 & 0 & 6 \\ \hline 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 3 & 1 & 8 \end{array}\right]$$

Optimal? Yes.

Back to the Example

$$\begin{array}{c|rrr|r} z & x_1 & x_2 & e_1 & \text{rhs} \\ 1 & -1 & -1 & 0 & 0 \\ \hline 0 & 2 & 1 & -1 & 4 \\ 0 & 1 & 2 & 0 & 6 \end{array}$$

What if we had chosen x_2 and e_1 as our basic variables?

$$\left[\begin{array}{c|rcc|r} z & x_1 & x_2 & e_1 & \text{rhs} \\ 1 & -1/2 & 0 & 0 & 3 \\ \hline 0 & 1/2 & 1 & 0 & 3 \\ 0 & -3/2 & 0 & 1 & -1 \end{array}\right]$$

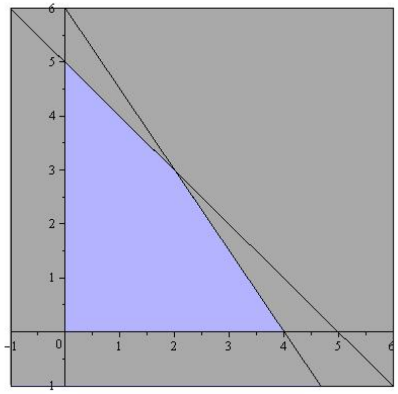
Conclusion?
Not every choice of basic variables will be feasible!

Example 4

$$\begin{array}{rll} \max_x & 6x_1 + & 5x_2 \\ \text{s.t.} & x_1 + & x_2 \le 5 \\ & 3x_1 + & 2x_2 \le 12 \\ & x_1, & x_2 \ge 0 \end{array}$$

$$\begin{array}{rrrr|r} -6 & -5 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 5 \\ 3 & 2 & 0 & 1 & 12 \end{array}$$

$$\begin{array}{rrrr|r} 0 & 0 & 3 & 1 & 27 \\ \hline 0 & 1 & 3 & -1 & 3 \\ 1 & 0 & -2 & 1 & 2 \end{array}$$



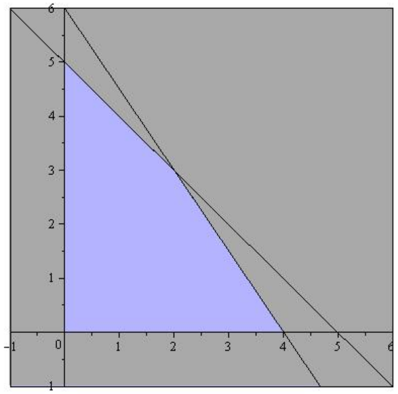
Solution is $(2, 3)$

Slight Change:

$$\begin{array}{lll} \max_x & 6x_1+ & 4x_2 \\ \text{s.t.} & x_1+ & x_2 \le 5 \\ & 3x_1+ & 2x_2 \le 12 \\ & x_1, & x_2 \ge 0 \end{array}$$

−6	−4	0	0	0
1	1	1	0	5
3	2	0	1	12

0	0	0	2	24
0	1/3	1	−1/3	1
1	2/3	0	1/3	4



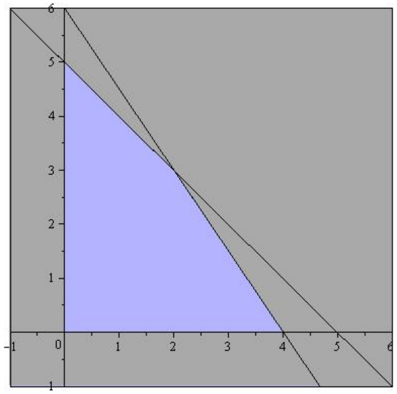
New solution is $(4, 0)$

We can bring in x_2 with no change to z:

0	0	0	2	24
0	1/3	1	−1/3	1
1	2/3	0	1/3	4

$$\Downarrow$$

0	0	0	2	24
0	1	3	−1	3
1	0	−2	1	2



New solution is $(2, 3)$. Any point on that line segment is also a maximizer.

Alternative Optimal Solutions

- If a NBV in Row 0 is 0, and we can pivot in this column (and maintain the same value of z), then we may have alternative optimal solutions.
- If two BFS are optimal, the line segment joining them is also optimal (by convexity).
- There may be "stalling" behavior. That is, it may be that once we pivot into the new BV, there may be a new way to proceed (for a better value of z).

Example

Consider the following "final" tableau:

z	x_1	x_2	x_3	x_4	rhs
1	0	0	0	2	2
0	1	0	−1	1	2
0	0	1	−2	3	3

Interpretation? Row 0 may have a 0 for x_3, but we cannot pivot in that column.

How many solutions to the optim. problem do we have?

Example

Consider the following LP:

$$\begin{array}{rl} \max & 4x_1 + 3x_2 \\ \text{st} & x_1 - 6x_2 \leq 5 \\ & 3x_1 \leq 11 \\ & x_1, x_2 \geq 0 \end{array}$$

Start as usual:

$$\begin{array}{rrrr|r} -4 & -3 & 0 & 0 & 0 \\ \hline 1 & -6 & 1 & 0 & 5 \\ 3 & 0 & 0 & 1 & 11 \end{array} \quad \Rightarrow \quad \begin{array}{rrrr|r} 0 & -3 & 0 & 4/3 & 44/3 \\ \hline 0 & -6 & 1 & -1/3 & 4/3 \\ 1 & 0 & 0 & 1/3 & 11/3 \end{array}$$

What's going on?

Write out the system of equations:

$$\begin{array}{rrrrr} \max & z = & 44/3 & +3x_2 & -(4/3)s_2 \\ \text{st} & x_1 = & 1/3 & & -(1/3)s_2 \\ & x_2 = & & x_2 & \\ & s_1 = & 4/3 & +6x_2 & +(1/3)s_2 \\ & s_2 = & & & s_2 \end{array}$$

How should we maximize this? Set $s_2 = 0$, and

$$\mathbf{x} = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 6 \\ 0 \end{bmatrix}$$

This increases the value of z without bound. Notice that $\mathbf{c}^T\mathbf{d} \neq 0$.

$$\begin{array}{rl} \min & z = -x_1 + 2x_2 \\ \text{st} & x_1 - x_2 \leq 1 \\ & x_1 - 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

Proceed as usual:

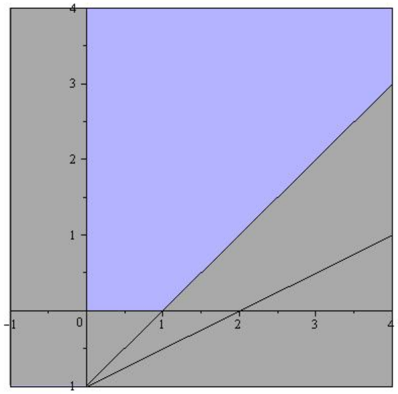
$$\begin{array}{rrrr|r} -1 & 2 & 0 & 0 & 0 \\ \hline 1 & -1 & 1 & 0 & 1 \\ 1 & -2 & 0 & 1 & 2 \end{array} \quad \Rightarrow \quad \begin{array}{rrrr|r} 0 & 1 & 1 & 0 & 1 \\ \hline 1 & -1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 \end{array}$$

Interpretation?

There is an optimal solution:

$$(1, 0)$$

The feasible set is unbounded.



Two Types of Unboundedness

- The objective function is unbounded (as is the feasible region).
- The feasible region is unbounded, but the objective function is not.

  "The LP is unbounded if there is a negative coefficient in Row 0, and all the remaining elements in the column are negative or zero"