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  4. Repeat.

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Maximizer: $x_1 = 2, x_2 = 3$ and $s_1 = s_2 = 0$. Notice also the top row:

$$z = 27 - 3s_1 - s_2$$

From Wed: Unbounded LP

Here was the final tableau for an unbounded region and an unbounded LP:

	z	x_1	x_2	x_3	s_1	s_2	s_3	rhs
	1	1	0	0	0	3	-2	50
s_1	0	4	0	0	1	-1	0	5
x_2	0	1	1	0	0	1	-2	10
x_3	0	0	0	1	0	1	-1	15

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Translation: $\max z = 50 - x_1 - 3s_2 + 2s_3$ with

$$\mathbf{x} = \begin{bmatrix} 0 \\ 10 \\ 15 \\ 5 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example 3, A Little Foreshadowing

$$\begin{aligned} \max z = \quad & x_1 & +x_2 & & \\ \text{st} \quad & 2x_1 & +x_2 & \geq 4 \\ & x_1 & +2x_2 & = 6 \\ & x_1, & x_2 & \geq 0 \end{aligned}$$

Build the initial tableau.

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\end{array}
$$

Build the initial tableau.

z	x_1	x_2	e_1	rhs
1	-1	-1	0	0
0	2	1	-1	4
0	1	2	0	6

Not standard form... No initial BFS.

Choosing x_1, x_2 as the basic variables, we can perform the row reduction using an inverse matrix:

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The new tableau:

$$\left[\begin{array}{c|rrr|r} z & x_1 & x_2 & e_1 & \text{rhs} \\ 1 & 0 & 0 & -1/3 & 10/3 \\ \hline 0 & 1 & 0 & -2/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & 8/3 \end{array}\right]$$

Bring e_1 into the set of BV (take x_2 out):

$$\left[\begin{array}{c|ccc|c} z & x_1 & x_2 & e_1 & \text{rhs} \\ 1 & 0 & 1 & 0 & 6 \\ \hline 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 3 & 1 & 8 \end{array}\right]$$

Optimal?

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Optimal? Yes.

Back to the Example

z	x_1	x_2	e_1	rhs
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What if we had chosen x_2 and e_1 as our basic variables?

Back to the Example

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\begin{array}{c|rrr|r}
z & x_1 & x_2 & e_1 & \text{rhs} \\
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\hline
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$$

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$$
\left[
\begin{array}{c|rrr|r}
z & x_1 & x_2 & e_1 & \text{rhs} \\
1 & -1/2 & 0 & 0 & 3 \\
\hline
0 & 1/2 & 1 & 0 & 3 \\
0 & -3/2 & 0 & 1 & -1
\end{array}
\right]
$$

Conclusion?

Back to the Example

$$\begin{array}{c|ccc|c}
z & x_1 & x_2 & e_1 & \text{rhs} \\
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\hline
0 & 1/2 & 1 & 0 & 3 \\
0 & -3/2 & 0 & 1 & -1
\end{array}\right]$$

Conclusion?

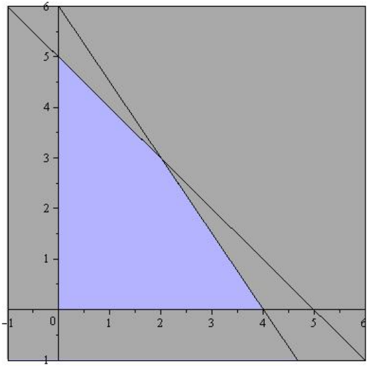
Not every choice of basic variables will be feasible!

Example 4

$$\begin{array}{rlll} \max_x & 6x_1 + & 5x_2 & \\ \text{s.t.} & x_1 + & x_2 & \leq 5 \\ & 3x_1 + & 2x_2 & \leq 12 \\ & x_1, & x_2 & \geq 0 \end{array}$$

$$\begin{array}{rrrr|r} -6 & -5 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 5 \\ 3 & 2 & 0 & 1 & 12 \end{array}$$

$$\begin{array}{rrrr|r} 0 & 0 & 3 & 1 & 27 \\ \hline 0 & 1 & 3 & -1 & 3 \\ 1 & 0 & -2 & 1 & 2 \end{array}$$



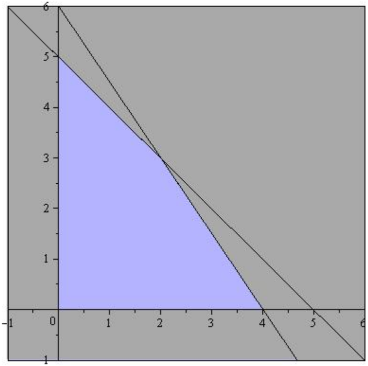
Solution is $(2, 3)$

Slight Change:

$$\begin{array}{rll} \max_x & 6x_1 + & 4x_2 \\ \text{s.t.} & x_1 + & x_2 \le 5 \\ & 3x_1 + & 2x_2 \le 12 \\ & x_1, & x_2 \ge 0 \end{array}$$

$$\begin{array}{rrrr|r} -6 & -4 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 5 \\ 3 & 2 & 0 & 1 & 12 \end{array}$$

$$\begin{array}{rrrr|r} 0 & 0 & 0 & 2 & 24 \\ \hline 0 & 1/3 & 1 & -1/3 & 1 \\ 1 & 2/3 & 0 & 1/3 & 4 \end{array}$$



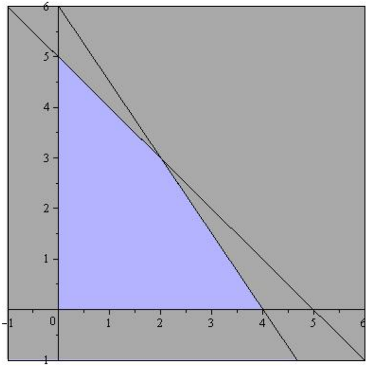
New solution is $(4, 0)$

We can bring in x_2 with no change to z:

$$\begin{array}{cccc|c} 0 & 0 & 0 & 2 & 24 \\ \hline 0 & 1/3 & 1 & -1/3 & 1 \\ 1 & 2/3 & 0 & 1/3 & 4 \end{array}$$

$$\Downarrow$$

$$\begin{array}{cccc|c} 0 & 0 & 0 & 2 & 24 \\ \hline 0 & 1 & 3 & -1 & 3 \\ 1 & 0 & -2 & 1 & 2 \end{array}$$



New solution is $(2,3)$. Any point on that line segment is also a maximizer.

Alternative Optimal Solutions

- If a NBV in Row 0 is 0, and we can pivot in this column (and maintain the same value of z), then we may have alternative optimal solutions.

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Alternative Optimal Solutions

- If a NBV in Row 0 is 0, and we can pivot in this column (and maintain the same value of z), then we may have alternative optimal solutions.
- If two BFS are optimal, the line segment joining them is also optimal (by convexity).
- There may be "stalling" behavior. That is, it may be that once we pivot into the new BV, there may be a new way to proceed (for a better value of z).

Example

Consider the following “final” tableau:

z	x_1	x_2	x_3	x_4	rhs
1	0	0	0	2	2
0	1	0	-1	1	2
0	0	1	-2	3	3

Interpretation?

Example

Consider the following “final” tableau:

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Interpretation? Row 0 may have a 0 for x_3, but we cannot pivot in that column.

How many solutions to the optim. problem do we have?

Example

Consider the following LP:

$$\begin{array}{rl} \max & 4x_1 + 3x_2 \\ \text{st} & x_1 - 6x_2 \leq 5 \\ & 3x_1 \leq 11 \\ & x_1, x_2 \geq 0 \end{array}$$

Start as usual:

$$\begin{array}{rrrr|r} -4 & -3 & 0 & 0 & 0 \\ \hline 1 & -6 & 1 & 0 & 5 \\ 3 & 0 & 0 & 1 & 11 \end{array} \quad \Rightarrow \quad \begin{array}{rrrr|r} 0 & -3 & 0 & 4/3 & 44/3 \\ \hline 0 & -6 & 1 & -1/3 & 4/3 \\ 1 & 0 & 0 & 1/3 & 11/3 \end{array}$$

What's going on?

Write out the system of equations:

$$\begin{array}{rrcrrr}
\max & z = & 44/3 & +3x_2 & -(4/3)s_2 \\
\text{st} & x_1 = & 1/3 & & -(1/3)s_2 \\
& x_2 = & & x_2 & \\
& s_1 = & 4/3 & +6x_2 & +(1/3)s_2 \\
& s_2 = & & & s_2
\end{array}$$

How should we maximize this?

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& x_2 = & & x_2 & \\
& s_1 = & 4/3 & +6x_2 & +(1/3)s_2 \\
& s_2 = & & & s_2
\end{array}$$

How should we maximize this? Set $s_2 = 0$, and

$$\mathbf{x} = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 6 \\ 0 \end{bmatrix}$$

This increases the value of z without bound. Notice that $\mathbf{c}^T\mathbf{d} \neq 0$.

$$\begin{aligned} \min \quad & z = -x_1 + 2x_2 \\ \text{st} \quad & x_1 - x_2 \leq 1 \\ & x_1 - 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Proceed as usual:

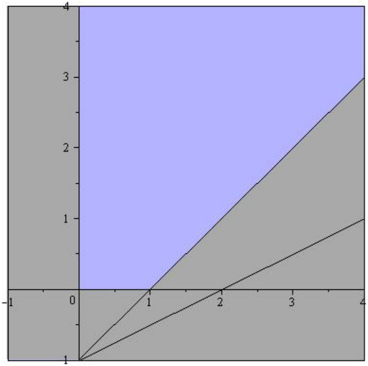
$$\begin{array}{rrrr|r} -1 & 2 & 0 & 0 & 0 \\ \hline 1 & -1 & 1 & 0 & 1 \\ 1 & -2 & 0 & 1 & 2 \end{array} \quad \Rightarrow \quad \begin{array}{rrrr|r} 0 & 1 & 1 & 0 & 1 \\ \hline 1 & -1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 \end{array}$$

Interpretation?

There is an optimal solution:

$$(1, 0)$$

The feasible set is unbounded.



Two Types of Unboundedness

- The objective function is unbounded (as is the feasible region).
- The feasible region is unbounded, but the objective function is not.

"The LP is unbounded if there is a negative coefficient in Row 0, and all the remaining elements in the column are negative or zero"