Solve the linear first order system $\mathbf{x}' = A\mathbf{x}$

Given

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x_1' \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x}' = A\mathbf{x}$$

Then we could solve the system in three ways:

- Convert the system to 2d order DE of the form $ay'' + by' + cy = 0$, and solve.
- Could try to write the system in implicit form:

  $$\frac{dx_2}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{cx_1 + dx_2}{ax_1 + bx_2}$$

  Then we could try to use a first order method.
- Use the eigenvalues and eigenvectors from the matrix A. This is summarized below for 2×2 matrices.

Eigenvalues and Eigenvectors

- Definition: Given an $n \times n$ matrix A, if there is a constant λ and a non-zero vector $\mathbf{v}$ so that

  $$A\mathbf{v} = \lambda\mathbf{v}$$

  then λ is an eigenvalue, and $\mathbf{v}$ is an associated eigenvector for matrix A. Note that an eigenvector is not uniquely determined; we usually choose the simplest vector (integer valued if possible).
- If we try to solve our equation:

  $$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \qquad \text{or} \qquad (A - \lambda I)\mathbf{v} = \mathbf{0} \tag{1}$$

  This system has a non-trivial (non-zero) solution for v_1, v_2 only if the determinant is zero (so that $A - \lambda I$ is not invertible):

  $$|A - \lambda I| = 0$$

  And this is the **characteristic equation**, and simplifying it will give you an n^{th} degree polynomial in λ, which we solve for the eigenvalues.

  If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the characteristic equation becomes:

  $$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \quad \Leftrightarrow \lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$$

  where $\text{Tr}(A)$ is the trace of A (which we defined as $a + d$). For each λ, we must go back and solve Equation (1).

- Given a λ and $\mathbf{v}$, the generalized eigenvector $\mathbf{w}$ is computed as the solution to

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

and in the 2×2 case, it is used when λ has algebraic multiplicity 2, but geometric multiplicity 1 (a double root with only one eigenvector).

Summary

To solve $\mathbf{x}' = A\mathbf{x}$, find the trace, determinant and discriminant for the matrix A. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on Δ:

- Real λ_1, λ_2 give two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \mathrm{Real}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Imag}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector $\mathbf{v}$. Compute the generalized eigenvector $\mathbf{w}$, $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(C_1\mathbf{v} + C_2\left(t\mathbf{v} + \mathbf{w}\right)\right)$$

You might find this helpful- Below there is a chart comparing the solutions from Chapter 3 to the solutions in Chapter 7:

	Chapter 3	Chapter 7
Form:	$ay'' + by' + cy = 0$	$\mathbf{x}' = A\mathbf{x}$
Ansatz:	$y = \mathrm{e}^{rt}$	$\mathbf{x} = \mathrm{e}^{\lambda t}\mathbf{v}$
Char Eqn:	$ar^2 + br + c = 0$	$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$
Real Solns	$y = C_1\mathrm{e}^{r_1 t} + C_2\mathrm{e}^{r_2 t}$	$\mathbf{x}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$
Complex	$y = C_1\mathrm{Re}(\mathrm{e}^{rt}) + C_2\mathrm{Im}(\mathrm{e}^{rt})$	$\mathbf{x}(t) = C_1\mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2\mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$
Repeated Root	$y = \mathrm{e}^{rt}(C_1 + C_2 t)$	$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(C_1\mathbf{v} + C_2\left(t\mathbf{v} + \mathbf{w}\right)\right)$

Exercise Set 3

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1\mathrm{e}^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2\mathrm{e}^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\mathbf{x}$$

2. For each matrix, find the eigenvalues and eigenvectors (these are selected from 16-23, p. 384 in the textbook). Note that they could be complex, and the matrix A may have complex numbers. Try the last one to see if you can do it!

(a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

(f) $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$

3. For each given λ and $\mathbf{v}$, find an expression for $\mathrm{Re}(e^{\lambda t}\mathbf{v})$ and $\mathrm{Im}(e^{\lambda t}\mathbf{v})$:

   (a) $\lambda = 3i$, $\mathbf{v} = [1-i, 2i]^T$

   (b) $\lambda = 1+i$, $\mathbf{v} = [i, 2]^T$

   (c) $\lambda = 2-i$, $\mathbf{v} = [1, 1+2i]^T$

   (d) $\lambda = i$, $\mathbf{v} = [2+3i, 1+i]^T$

4. Given the eigenvalues and eigenvectors for some matrix A, write the general solution to $\mathbf{x}' = A\mathbf{x}$. Furthermore, classify the origin as a sink, source, saddle, or none of the above.

   (a) $\lambda = -2, 3 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (b) $\lambda = -2, -2 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

   (c) $\lambda = 2, -3 \qquad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

5. Give the general solution to each system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, and sketch a phase plane (solutions in the x_1, x_2 plane). Identify the origin as a *sink*, *source* or *saddle*:

   (a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -6 & 10 \\ -2 & 3 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 8 & 6 \\ -15 & -11 \end{bmatrix}$

6. (Extra Practice) For each system below, find y as a function of x by first writing the differential equation as dy/dx.

   (a) $\begin{aligned} x' &= -2x \\ y' &= y \end{aligned}$

   (b) $\begin{aligned} x' &= y + x^3 y \\ y' &= x^2 \end{aligned}$

   (c) $\begin{aligned} x' &= -(2x+3) \\ y' &= 2y - 2 \end{aligned}$

   (d) $\begin{aligned} x' &= -2y \\ y' &= 2x \end{aligned}$