Review 1, Mathematical Modeling

In this part of the modeling course, we have looked at building models by theory- That is, we build differential equations (or difference equations) with specific assumptions about the phenomenon being modeled.

Part of the modeling course is building the model, but just as importantly, we need to be able to analyze our model in order to see if it properly captures what we think it ought to capture, and to see if it reflects that part of nature that we wish to capture in the model.

This exam will be all in-class and will be 50 minutes in length. Generally speaking, the exam will cover topics from Chapters 7, 9 and section 2.9 from the Boyce and Diprima text.

Topics from Chapter 7

Here the main idea is to be able to solve a linear first order system of differential equations. In particular, we looked at the details when A is 2×2:

$$\mathbf{x}' = A\mathbf{x} \quad \Rightarrow \quad \begin{bmatrix} x_1'(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(Typically, we supress the dependence on time). To solve this differential equation, we began with the ansatz: $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$, and we showed that λ had to be an eigenvalue of A, and $\mathbf{v}$ a corresponding eigenvector. Finding the eigenvalues means solving the characteristic equation:

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad - bc) = 0 \quad \Rightarrow \quad \lambda^2 - \mathrm{Tr(A)}\lambda + \det(A) = 0$$

We have three cases, dependent on the discriminant $\Delta = \mathrm{Tr}(A)^2 - 4\det(A)$.

Solutions to $\mathbf{x}' = A\mathbf{x}$

- If $\Delta > 0$, then we have two real, distinct eigenvalues. The general solution to the DE is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- If $\Delta < 0$, then we have complex conjugate eigenvalues. Using just $\lambda = \alpha + \beta i$, and the corresponding eigenvector, the general solution to the DE is:

$$\mathbf{x}(t) = C_1 \mathrm{Real}(\mathrm{e}^{\lambda t}\mathbf{v}) + C_2 \mathrm{Imag}(\mathrm{e}^{\lambda t}\mathbf{v})$$

- If $\Delta = 0$ AND the dimension of the eigenspace is 2, then go to the first case.

  If $\Delta = 0$ AND the dimension of the eigenspace is 1 (so the matrix is defective), then we compute a second vector known as a generalized eigenvector $\mathbf{w}$ that satisfies the equation:

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

  Then the solution to the DE is given by:

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(C_1\mathbf{v} + C_2(t\mathbf{v} + \mathbf{w})\right)$$

Classify the Equilibrium

We have used the Poincaré Diagram to classify the equilibrium solutions. I will provide that during the exam as well.

Other notes about systems

- We should be able to convert any second order differential equation into an equivalent system of first order, and most systems of first order into an equivalent differential equation of second order.
- We can also solve a system of first order differential equations using the techniques we learned to solve a second order differential equation ("Chapter 3 techniques" outlined in the notes).
- We can solve a system of first order equations by first creating dy/dx, then solving using our usual first order methods (first order linear equation with an integrating factor, separable differential equation).

Topics from Chapter 9, Nonlinear Differential Equations

There are several possible techniques for solving nonlinear systems of differential equations; for example, given the system $(x'(t), y'(t))$, we might be able to integrate the equation that we get for dy/dx.

Our goal was to get a more general technique:

Analysis by Local Linearization

First, we defined what we meant by local linearization. Generally, the idea is to replace the nonlinear function f by a linear function, based at some point (our point will be an equilibrium solution).

In the most general case, the linearization is given below at a point $\mathbf{x} = \mathbf{a}$:

$$\mathbf{y} = \vec{F}(\mathbf{x}) \quad \Rightarrow \quad L(\mathbf{x}) = \vec{F}(\mathbf{a}) + D\vec{F}(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

Written out using the coordinate functions:

$$\vec{F}(\mathbf{x}) \begin{bmatrix} f_1(x_1, \cdots, x_n) \\ f_2(x_1, \cdots, x_n) \\ \vdots \\ f_m(x_1, \cdots, x_n) \end{bmatrix} \quad \Rightarrow \quad L(\mathbf{x}) = \begin{bmatrix} f_1(a_1, \cdots, a_n) \\ f_2(a_1, \cdots, a_n) \\ \vdots \\ f_m(a_1, \cdots, a_n) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{bmatrix}$$

The matrix is a matrix of first partial derivatives (called the Jacobian matrix), and here it has been evaluated at the point $\mathbf{x} = \mathbf{a}$, so this is a matrix of constants, not formulas!

Local Linear Analysis

Given a nonlinear first order system of differential equations:

1. Find all the equilbrium solutions: $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k$.
2. About each equilibrium:
   - Linearize the system
   - Use the Poincaré Diagram to classify the equilbrium.
3. Try to get a global picture as to what is happening. Sometimes it is helpful to get a direction field (or slope field) to clarify.

Models we have considered

- **The SIR model.** Given $s(t), i(t), r(t)$ as the susceptible, infected and recovered proportions of the population at time t, we constructed the model of sickness as:

$$\begin{aligned} s' &= -bsi \\ i' &= bsi - ki \\ r' &= ki \end{aligned}$$

  You should be able to discuss the expressions you see in the DE in terms of the physical situation (for example, what is the significance of having an "si" term in the first equation? What happens when you add the three equations together- What is being assumed there?

- **Tank Mixing:** Recall that the rate of change of the quantity is modeled as "Rate In − Rate Out", and be sure you're making the units match up.

  We get a system of first order if we have more than one tank.

  - This exercise is also where we learned how to convert a differential equation of the form:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}$$

    into a differential equation of the form: $\mathbf{u}' = A\mathbf{u}$. You'll notice that this is the DE we solve in Chapter 7.

- **Population Models:** Population models are excellent template models to know. From Math 244, we have the *exponential growth model*, $y' = ky$, and the *logistic growth model*, $y' = ky(b - y)$, which incorporates an environmental threshold. We reviewed how to analyze these models using the (y, y') graph to find and classify the equilibrium solutions.

- **Competition Between Species**

- **Predator-Prey Models**

Discrete Dynamical Systems (Difference Equations)

Topic list:

- Vocabulary: Orbit, cobweb diagram, solution to a difference equation,
- Be able to find the equilbrium solutions (also known as the "fixed points" or "critical points").
- Graphically compute the orbit using a cobweb diagram. Identify the equilibria from a cobweb diagram.
- Give the solution to certain types of difference equations:
  1. From first principles (like Exercises 1-6 in Section 2.9 of Boyce and Diprima)
  2. $y_{n+1} = \alpha y_n$

     In this case, the solution is: $y_n = (y_0)\alpha^n$
  3. $y_{n+1} = \alpha y_n + b$

     In this case, we can write the solution as either:

     $$y_n = \alpha^n (y_0 - y_E) + y_E \qquad \text{or } y_n = c \cdot \alpha^n + \frac{1 - \alpha^n}{1 - \alpha} b$$

     You should note that the expression in front of b is the sum of $1 + a + a^2 + \cdots + a^{n-1}$ from a geometric series.
  4. $y_{n+1} = \alpha y_n + b(n)$, when $b(n)$ is a polynomial in n.

     The homogeneous part of the solution is $c \cdot \alpha^n$. For the particular part, guess that the solution is the same degree polynomial as $b(n)$. Substitute this into the difference equation, then solve for the coefficients of your polynomial.
  5. Linear second order difference equations: $ay_{n+2} + by_{n+1} + cy_n = 0$

     The ansatz we use is $y_n = \lambda^n$. With this, we get that λ must satisfy the characteristic equation, which in this case, is:

     $$a\lambda^2 + b\lambda + c = 0$$

     so that the form of the solution depends on the discriminant $\Delta = b^2 - 4ac$

     (a) If $\Delta > 0$, the solution is:

     $$y_n = C_1 \lambda_1^n + C_2 \lambda_2^n$$

     (b) If $\Delta = 0$, then the solution is:

     $$y_n = \lambda^n (C_1 + nC_2)$$

(c) If $\Delta < 0$, let $\lambda = \alpha + \beta i$.
First rewrite λ in polar form, $\lambda = re^{i\theta}$. Then we write the solution as:

$$y_n = C_1 \text{Real}(\lambda^n) + C_2 \text{Imag}(\lambda^n) = r^n (C_1 \cos(n\theta) + C_2 \sin(n\theta))$$

- Be able to convert a second order linear difference equation into a system of first order (like we did with differential equations), and vice versa.