Review Questions, Mathematical Modeling

1. Let $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 6 & -4 \end{bmatrix} \mathbf{x}$. Convert this system to an equivalent second order linear homogeneous differential equation, then solve that.

2. Let $y'' - 6y' + 9y = 0$ with $y(0) = 1$, $y'(0) = 2$. Convert this into an equivalent system of first order differential equations, then solve it using eigenvectors and eigenvalues.

3. Given each matrix A below, give the general solution to $\mathbf{x}' = A\mathbf{x}$, and classify the equilibrium as to its stability (you may use the Poincaré Diagram, if needed).

   (a) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & -17 \\ 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

4. Suppose we have brine pouring into tank A at a rate of 2 gallons per minute, and salt is in the brine at a concentration of 1/2 pound per gallon. Brine is being pumped into tank A from tank B (well mixed) at a rate of 1 gallon per minute. Brine is pumped out of tank A at a rate of 3 gallons per minute to tank B, and brine is poured into tank B from an external source at a rate of 2 gallons per minute, and 1/3 pound of salt per gallon. Initially, both tanks have 100 gallons of clear water.

   Write the system of differential equations that model the amount of salt in the tanks at time t.

5. Consider the system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ given below:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

   (a) Find the equilibrium solution, $\mathbf{x}_E$.

   (b) Show that, if $\mathbf{u} = \mathbf{x} - \mathbf{x}_E$, then the differential equation for $\mathbf{u}$ is: $\mathbf{u}' = A\mathbf{u}$.

   (c) Solve the differential equation by first solving the DE for $\mathbf{u}$.

6. Use the Poincaré Diagram to determine how the origin changes stability by changing α if

$$\mathbf{x}' = \begin{bmatrix} \alpha + 1 & \alpha \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

7. Let F be given below, and linearize it at the given value.

   (a) $\mathbf{F}(t) = \begin{bmatrix} t^2 + 3t + 2 \\ \sqrt{t+1} + 1 \\ \sin(t) \end{bmatrix}$ at $t = 0$

   (b) $f(x, y, z) = x^2 + 3x + 2y + 4z - 2$ at $(x, y, z) = (1, -1, 1)$

(c) $\mathbf{F}(x,y) = \begin{bmatrix} x^2+3xy-y+1 \\ y^2+2xy+x^2-1 \end{bmatrix}$ at $(x,y)=(1,0)$

8. For each nonlinear system below, perform a local linear analysis about all equilibria.

   (a) $\begin{aligned} dx/dt &= x - xy \\ dy/dt &= y + 2xy \end{aligned}$ (b) $\begin{aligned} dx/dt &= 1+2y \\ dy/dt &= 1-3x^2 \end{aligned}$

9. For each of the systems in question 8, solve them by first computing dy/dx.

10. For 8(a) above, if x and y were two populations, what kinds of assumptions are being made to result in these differential equations?

11. Given $x' = f(x,y)$ and $y' = g(x,y)$, then a **nullcline** is a curve where f or g is 0. Note that an equilibrium is where the nullclines intersect.

    If $x' = -4x + y + x^2$ and $y' = 1 - y$, then graph the nullclines, taking note of the equilbrium solutions. Is there an area in your drawing where $x' < 0$ and $y' < 0$? Make note of it.

12. Is the following system an example of predator-prey or competing species? In either case, perform a local linear analysis:

$$\begin{aligned} x' &= x(1-0.5y) \\ y' &= y(-0.75+0.25x) \end{aligned}$$

13. Solve:

    (a) $x_{n+1} = x_n + 1$

    (b) $x_{n+1} = 5x_n + n^2$

    (c) $x_{n+1} = \frac{1}{2}x_n + 3^n$

14. Assume the temperature of a roast in the oven increases at a rate proportional to the difference between the oven temperature (set to 400) and the roast temperature. If the roast enters the oven at 50 degrees, and is measured one hour later to be 90, when will the roast reach the FDA safe temperature of 160? (Hint: Write down, then solve the difference equation).

15. Convert the following system of difference equations to a second order difference equation, and solve it if $x_0 = y_0 = 1$.

$$\begin{aligned} x_{n+1} &= 2y_n \\ y_{n+1} &= 3x_n \end{aligned}$$

16. Solve the second order difference equation with $x_0 = 1$, $x_1 = -1$.

(a) $x_{n+2} - x_n = 0$

(b) $x_{n+2} + x_n = 0$

(c) $x_{n+2} + 3x_{n+1} + x_n = 0$

(d) $x_{n+2} = x_{n+1} + x_n$ (Do you recognize this famous difference equation? Typically we set $x_0 = 1, x_1 = 1$ in this example, so you can solve it that way.)

17. Consider the difference equation:

$$x_{n+2} + \alpha x_{n+1} + \beta x_n = 0$$

where we assume $\alpha^2 = 4\beta = 0$. Show that $x_n = n(-\alpha/2)^n$ is a solution.

18. Convert the following equations to equivalent systems of first order:

(a) $x_{n+2} + x_{n+1} - x_n = 0$

(b) $x_{n+2} + 3x_{n+1} + x_n = 0$

19. Convert the following systems of first order into an equivalent difference equation of second order.

(a) $\begin{aligned} a_{n+1} &= a_n + b_n \\ b_{n+1} &= 3a_n + b_n \end{aligned}$

(b) $\begin{aligned} a_{n+1} &= 2a_n + b_n \\ b_{n+1} &= a_n + 2b_n \end{aligned}$

20. Is it always possible to convert a system of two linear equations (as in the last problem) to a single second order difference equation? Explain.