The Important Stuff from 3.2 and 3.3

The last handout was a summary of how the theory of linear algebra and the theory of linear differential equations are intricately interwoven. While this is interesting and intriguing (and is continued in Chapter 4), we only had a short time to talk about it.

So, here is a brief summary of the most important definitions and theorems about Second Order, Linear, Homogeneous Differential Equations (SOLHDE, for you engineers!).

1. The Existence and Uniqueness Theorem.

   Let $y'' + p(t)y' + q(t)y = f(t)$, $y(t_0) = y_0$. Then if p, q, f are all continuous for $t \in [a, b]$, (and of course, $t_0 \in [a, b]$), then there exists a unique solution to the differential equation, and this solution persists for all $t \in [a, b]$.

2. Definitions:

   (a) Linear Independence: A set of functions, $\{y_1(t), \ldots, y_k(t)\}$ is said to be linearly independent on the interval $[a, b]$ iff the only solution to:

   $$c_1 y_1(t) + \ldots + c_k y_k(t) = 0$$

   (for all $t \in [a, b]$) is: $c_1 = c_2 = \ldots = c_k = 0$

   (b) The Wronskian of y_1 and y_2 is:

   $$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

   Note that this is a function of t.

   (c) The Fundamental Set of Solutions: y_1 and y_2 form a fundamental set of solutions to $y'' + p(t)y' + q(t)y = 0$ if an arbitrary solution can be written as a linear combination of y_1 and y_2. Thus, ALL solutions are of the form:

   $$c_1 y_1 + c_2 y_2$$

   if y_1 and y_2 form a fundamental set.

3. Theorem (Linear Independence and the Wronskian, in General)

   A set of functions is linearly independent on an interval $[a, b]$ if we can find a single value of t for which $W(y_1, \ldots, y_k)(t) \neq 0$.

4. Abel's Theorem (Linear Independence and the Wronskian, for Solutions to a Linear Homogeneous Differential Equation).

   If y_1 and y_2 are solutions to a linear second order homogeneous differential equation, then the Wronskian of y_1 and y_2 is either identically zero on an interval, or it is never zero on the interval.

5. Theorem (How many linearly independent solutions are there to an n^{th} order linear homogeneous differential equation?)

   There are exactly n linearly independent solutions to the n^{th} order linear homogeneous differential equation.