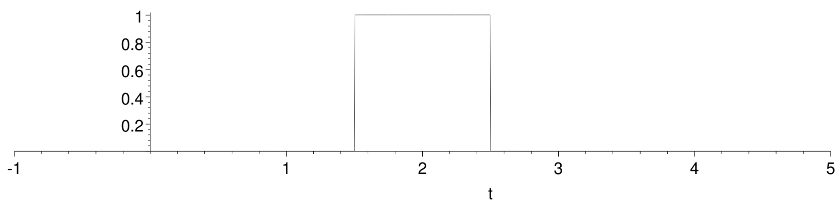
Problem 6.4, #16

$$u'' + \frac{1}{4}u' + u = kg(t), \quad u(0) = 0, u'(0) = 0$$

and $g(t) = u_{3/2}(t) - u_{5/2}(t)$.

1. Plot $g(t)$:



2. Solve for $u(t)$:

$$\begin{aligned}(s^2 + \frac{1}{4}s + 1)U &= \frac{k}{s}\left(e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}\right)\\ U &= \frac{1}{s(s^2 + \frac{1}{4}s + 1)} \cdot k\left(e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}\right)\end{aligned}$$

So, let

$$H(s) = \frac{1}{s(s^2 + \frac{1}{4}s + 1)}$$

Then the solution will be:

$$u(t) = k\left(u_{3/2}(t)h(t - \frac{3}{2}) - u_{5/2}(t)h(t - \frac{5}{2})\right)$$

Now,

$$\begin{aligned}H(s) &= \frac{1}{s(s^2 + \frac{1}{4}s + 1)}\\ &= \frac{1}{s} - \frac{s + \frac{1}{4}}{s^2 + \frac{1}{4}s + 1}\end{aligned}$$

$$= \frac{1}{s} - \frac{s + \frac{1}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}} - \frac{1}{3\sqrt{7}}\frac{\frac{3\sqrt{7}}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}}$$

so that

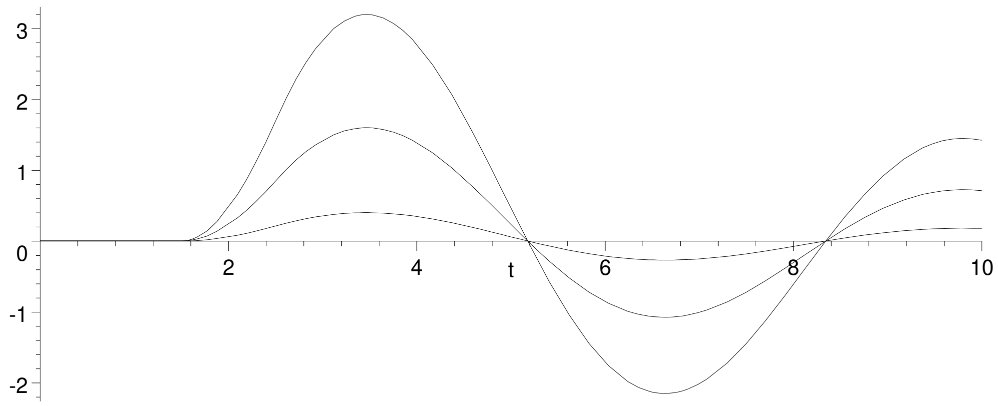
$$h(t) = 1 - e^{-\frac{1}{8}t}\cos\left(\frac{3\sqrt{7}t}{8}\right) - e^{-\frac{1}{8}t}\frac{1}{3\sqrt{7}}\sin\left(\frac{3\sqrt{7}t}{8}\right)$$

From the Maple document, we see that: (1) Changing the value of k only changes the amplitude of the pseudoperiodic solution, (2) The k for which $ku(t) = 2$ is approximately: $k \approx 2.5$, and (3) the time for which $2u(t) < 0.1$ for all $t > \tau$ is approximately $\tau \approx 25.6$

```
> u:=t->Heaviside(t):
> a:=3*sqrt(7)/8:
> b:=1/(8*a):
> h:=t->1-exp(-(1/8)*t)*(cos(a*t)+b*sin(a*t)):
> y:=t->u(t-(3/2))*h(t-(3/2))-u(t-(5/2))*h(t-(5/2));
```

$$y := t \to \mathrm{u}\left(t-\frac{3}{2}\right)\mathrm{h}\left(t-\frac{3}{2}\right)-\mathrm{u}\left(t-\frac{5}{2}\right)\mathrm{h}\left(t-\frac{5}{2}\right)$$

```
> plot([(1/2)*y(t),2*y(t),4*y(t)],t=0..10,color=[black,black,black])
  ;
```



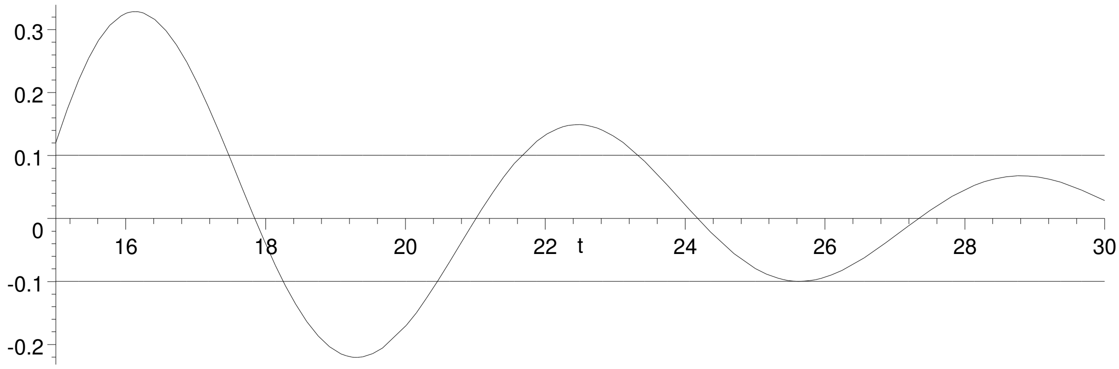

```
> s1:=fsolve(diff(y(t),t)=0,t=3..4);
```

$$s1 := 3.467484039$$

```
> k:=evalf(2/y(s1));
```

$$k := 2.502413956$$

```
> plot([2*y(t),0.1,-0.1],t=15..30,color=black);
```

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> fsolve(2*y(t)=-0.1,t=25..26);
```

$$25.58786296$$

Problem 6.5 #14

$$y'' + \gamma y' + y = \delta(t-1), \quad y(0) = 0, y'(0) = 0$$

We are initially asked to solve with $\gamma = \frac{1}{2}$, but it might be more instructive to give the general solution:

$$(s^2 + \gamma s + 1)Y = \mathrm{e}^{-s}$$
$$Y = \frac{1}{s^2 + \gamma s + 1}\mathrm{e}^{-s}$$

So, if $H(s) = \frac{1}{s^2+\gamma s+1}$, then:

$$\frac{1}{s^2 + \gamma s + 1} = \frac{1}{(s + \frac{\gamma}{2})^2 + (1 - \frac{\gamma^2}{4})} = \frac{2}{\sqrt{4-\gamma^2}} \frac{\frac{\sqrt{4-\gamma^2}}{2}}{(s + \frac{\gamma}{2})^2 + (1 - \frac{\gamma^2}{4})}$$

so that

$$h(t) = \frac{2}{\sqrt{4-\gamma^2}}\mathrm{e}^{-\frac{\gamma}{2}t} \sin\left(\frac{\sqrt{4-\gamma^2}}{2}t\right)$$

and the general solution is:

$$y(t) = u_1(t)h(t-1)$$

We can also solve for the maximum analytically. Recall that $u_1(t)h(t-1)$ is the graph of $h(t)$ shifted forward one unit, with the function being zero for $t < 1$. Therefore we'll solve for the maximum of $h(t)$. To simplify notation, let:

$$a = \frac{\sqrt{4-\gamma^2}}{2}, \quad b = \frac{\gamma}{2}$$

so that

$$h(t) = \frac{1}{a}\mathrm{e}^{-bt}\sin(at), \text{ and solving } h'(t) = \mathrm{e}^{-bt}\left(\frac{-b}{a}\sin(at) + \cos(at)\right) = 0$$

gives the maximum at:

$$t^* = \frac{1}{a}\tan^{-1}(a/b) \text{ which translates to } t^* = \frac{2}{\sqrt{4-\gamma^2}}\tan^{-1}\left(\frac{\sqrt{4-\gamma^2}}{\gamma}\right)$$

so that the original problem will have a maximum at:

$$t^* = \frac{2}{\sqrt{4-\gamma^2}}\tan^{-1}\left(\frac{\sqrt{4-\gamma^2}}{\gamma}\right) + 1$$

For part (a), $\gamma = \frac{1}{2}$, so

$$h(t) = \frac{4}{\sqrt{15}}\mathrm{e}^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right), \text{ with } t^* = 1 + \frac{4}{\sqrt{15}}\tan^{-1}(\sqrt{15}) \approx 2.3613 \text{ and } y(t^*) \approx 0.73$$

For part (c), we get, for $\gamma = \frac{1}{4}$,

$$h(t) = \frac{8}{3\sqrt{7}}\mathrm{e}^{-\frac{1}{8}t}\sin\left(\frac{3\sqrt{7}}{8}t\right), \text{ with } t^* = \frac{8}{3\sqrt{7}}\tan^{-1}(3\sqrt{7}) + 1 \approx 2.46, \text{ and } y(t*) \approx 0.834$$

And, as $\gamma \to 0$, we see that:

$$y(t) = u_1(t)\sin(t-1), \text{ with } t^* = 1 + \frac{\pi}{2}, \text{ and } y(t^*) = 1.$$

(You could've used Maple plots and numerical solvers for this, too).

Problems #10, #14, Section 6.6

#10

$$F(s) = \frac{1}{(s+1)^2(s^2+1)} = \frac{1}{(s+1)^2} \cdot \frac{1}{s^2+4}$$

so

$$f(t) = t\mathrm{e}^{-t} * \frac{1}{2}\sin(2t) = \frac{1}{2}\int_0^t (t-\lambda)\mathrm{e}^{-(t-\lambda)}\sin(2\lambda)\,d\lambda$$

#14

$$\begin{aligned}
4y'' + 4y' + 17y = &\quad g(t), \quad y(0) = 0, y'(0) = 0 \\
(4s^2 + 4s + 17)Y = &\quad G(s) \\
Y = &\quad \frac{1}{4} \cdot \frac{1}{s^2 + s + \frac{17}{4}} \cdot G(s) \\
= &\quad \frac{1}{4} \cdot \frac{1}{(s+\frac{1}{2})^2 + 4} \cdot G(s) \\
= &\quad \frac{1}{8} \cdot \frac{2}{(s+\frac{1}{2})^2 + 4} \cdot G(s) \\
y(t) = &\quad \frac{1}{8}\mathrm{e}^{-\frac{1}{2}t}\sin(2t) * g(t) \\
= &\quad \frac{1}{8}\int_0^t \mathrm{e}^{-\frac{1}{2}(t-\lambda)}\sin(2(t-\lambda))g(\lambda)d\,\lambda
\end{aligned}$$