KEY

Math 125: Exam the Second

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This exam is closed book, closed notes, closed colleague. You have until 9:55 to finish. **READ ALL INSTRUCTIONS CAREFULLY.** Please read the statement below and sign it when you are finished.

I have not used my calculator on this examination except for arithmetic, trigonometric, logarithmic, and exponential functions. I certify that the work on this exam is my own and that I have not discussed the specific contents of this exam with anyone prior to my taking it.

Signature:

1. Find $f'(x)$ in each case.

   (a) $f(x) = \cos(x)\sin(x)$

   $f'(x) = \cos x(\cos x) - \sin x(\sin x) = \cos^2 x - \sin^2 x$

   (b) $f(x) = (x^3 + 2x^2 + 1)^3$

   $f'(x) = 3(x^3+2x^2+1)^2 \cdot (3x^2+4x)$

   (c) $f(x) = \sqrt{2x - x^2}$ $= (2x-x^2)^{1/2}$

   $f'(x) = \frac{1}{2}(2x-x^2)^{-1/2} \cdot (2-2x)$

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   (d) $f(x) = 2^x \cdot 3^x$

   $f'(x) = 2^x \cdot 3^x \ln 3 + 3^x \cdot 2^x \ln 2 = 2^x 3^x (\ln 2 + \ln 3)$ OR $\frac{d}{dx} 6^x = 6^x \ln 6$

   (e) $f(x) = \arctan(x^2)$

   $f'(x) = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$

   (f) $f(x) = x^{2x}$

   $y = x^{2x}$ $\quad \ln y = 2x \ln x$

   $\frac{1}{y}\frac{dy}{dx} = 2x \cdot \frac{1}{x} + 2\ln x$, $\frac{dy}{dx} = (2+2\ln x)y = (2+2\ln x)x^{2x}$

2. If $x^2 + y^2 = 2 + \cos(y)$, find $\frac{dy}{dx}$.

diff. both sides w.r.t. x

$$2x + 2y \frac{dy}{dx} = 0 + (-\sin y) \frac{dy}{dx}$$

$$(2y + \sin y) \frac{dy}{dx} = -2x$$

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$$\frac{dy}{dx} = \frac{-2x}{2y + \sin y}$$

3. (a) What is $\frac{d}{dx} \ln(x)$?

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

(b) Determine $\frac{d}{dx} \ln(x^2)$ and $\frac{d}{dx} \ln(x^3)$.

$$\frac{d}{dy} \ln(x^2) = \frac{1}{x^2} \cdot 2x$$

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$$\frac{d}{dx} \ln(x^3) = \frac{1}{x^3} \cdot 3x^2$$

(c) Simplify your answers in part (b) as much as possible, and relate your three answers to one another using rules of logarithms.

$$\frac{d}{dx} \ln(x^2) = \frac{2}{x} \qquad \frac{d}{dx} \ln(x^3) = \frac{3}{x}$$

$$\frac{d}{dx}(\ln x^n) = \frac{n}{x}$$ as $\ln(x^n) = n \ln x$

so $\frac{d}{dx}(n \ln x) = n \cdot \frac{1}{x}$.

15 each

I will score your best THREE of the following FOUR problems. You may attempt all of the problems if you wish.

4. We are saving for our daughter's college education at a rate of \$300 per month, or \$3600 per year. The money we save is gaining interest at a rate of 5%. Additionally, her generous grandparents have given her \$10,000 to start the account.

   (a) The differential equation

   $$\frac{dP}{dt} = A + rP$$

   where P represents the principal in the savings account. (t is measured in years). Find values for A and r?

   $A \to$ money in per year $= 3600$

   $r \to$ rate $= .05$

   $$\frac{dP}{dt} = 3600 + .05P$$

   (b) Show that

   $$P(t) = \cancel{82000e^{.05t} - 72000} \quad 82000e^{.05t} - 72000$$

   satisfies the scenario by calculating $P(0)$, then differentiating and rewriting the derivative in terms of P to verify your answer to (a).

   $P(0) = 82{,}000\, e^{0} - 72000 = 82{,}000 - 72{,}000 = 10{,}000$ ✓

   $\frac{dP}{dt} = .05(82000\, e^{.05t})$

   $= .05(P + 72000) = .05P + 3600$ ✓

   (c) How much money will we have saved when she turns 18?

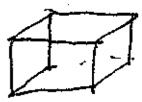
   $P(18) = 82{,}000\, e^{.05(18)} - 72{,}000 = 129{,}687.46.$

5. A box is in the shape of a cube. Its side length is growing at a rate of .5 inches/minute. How fast is the volume increasing when the side length is 4 inches? How fast is the surface area increasing at that same time?

6. We are boiling an egg in $100°C$ water. When we put the egg in the water, the temperature is $20°C$. One minute later, the egg is at $30°C$. We must cook the egg to *between* $75°C$ and $80°C$. Give the minimum and maximum cooking time for the egg.

7. Find the tangent line to $f(x) = x\sqrt{x}$ at $x = 9$ and use it to approximate $f(9.1)$.

5. $V = s^3$



$SA = 6s^2$

$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$

$\frac{dA}{dt} = 12s \frac{ds}{dt}$

$\frac{dV}{dt} = 3(16)(.5) = 24 \text{ in}^3/\text{min}$

$\frac{dA}{dt} = 12(4)(.5) = 24 \text{ in}^3/\text{min}$

6. $T = T_a + T_d e^{kt}$

$T_a = 100°C$

$T_d = 20 - 100 = -80°C$

$T = 100 - 80e^{kt}$

$T(1) = 100 - 80e^{k(1)} = 30$

$e^k = \frac{-70}{-80}$ $\quad k = \ln\frac{7}{8} = -.13353$

Solve for t:

$75 = 100 - 80e^{-.13353t}$ $\quad t = \ln\frac{-25}{-80} / k = 8.711$ minutes

$80 = 100 - 80e^{-.13353t}$ $\quad t = \ln\frac{-20}{80} / k = 10.382$ minutes

7. $f(x) = x\sqrt{x} = x^{3/2}$ point $(9, 27)$

$f(9) = 9\sqrt{9} = 9 \cdot 3 = 27$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$ $\quad f'(9) = \frac{3}{2}\sqrt{9} = \frac{9}{2}$ = slope

line: $y - 27 = \frac{9}{2}(x - 9)$ $\quad f(9.1) \approx 27 + \frac{9}{2}(9.1 - 9)$

$= 27 + .45 = 27.45$