1. (a) Suppose that

$$f(x) = \int_2^x \cot(t^3) + 2t\ dt$$

Find $f'(x)$.

SOLUTION: This is an application of the Fundamental Theorem of Calculus. $\int_2^x \cot(t^3) + 2t\ dt = F(x) - F(2)$ where F is an antiderivative of $\cot(t^3) + 2t$. Differentiating the integral with respect to x gives

$$\frac{d}{dx} \int_2^x \cot(t^3) + 2t\ dt = \cot(x^3) + 2x;$$

differentiation will eliminate the $F(2)$, as it is constant.

(b) Find $f'(x)$ if

$$f(x) = \int_2^{x^2} \cot(t^3) + 2t\ dt$$

SOLUTION: This is the same as part (a), only now we need to use the chain rule since we're differentiating $F(x^2) - F(2)$, in this case

$$\frac{d}{dx} \int_2^{x^2} \cot(t^3) + 2t\ dt = (\cot(x^6) + 2x^2) \cdot 2x$$

2. (a) Find

$$\int_3^{10} \frac{x}{x^2 - 4}\ dx$$

SOLUTION: This is a straightforward u-substitution. (Note that integration by partial fractions works as well, but is much ickier). Let $u = x^2 - 4$, $du = 2xdx$, $ll = 3^2 - 4 = 5$, $ul = 10^2 - 4 = 96$. Thus

$$\int_3^{10} \frac{x}{x^2 - 4}\ dx = \frac{1}{2} \int_5^{96} \frac{du}{u} = \frac{1}{2} \ln(u)|_5^{96} = \frac{1}{2}(\ln(96) - \ln(5))$$

(b) Discuss

$$\int_{-1}^{1} \frac{x}{x^2 - 4}\ dx$$

SOLUTION: The function being integrated is the quotient of an odd function and an even function and, hence, is odd (check that $f(-x) = -f(x)$). Since it is odd, and the bounds are from 1 to -1, the integral equals 0.

(c) Discuss

$$\int_0^5 \frac{x}{x^2 - 4}\ dx$$

SOLUTION: The interval of integration includes the value $x = 2$, for which the function is not defined. Hence, the integral is improper and needs to be split at 2. We split it as

$$\lim_{b \to 2^-} \int_0^2 \frac{x}{x^2 - 4}\, dx + \lim_{b \to 2^+} \int_b^5 \frac{x}{x^2 - 4}\, dx$$

We can show that each of these integrals does not approach a limit as x gets close to 2, hence, these integrals do not exist.

(d) Discuss

$$\int_4^\infty \frac{x}{x^2 - 4}\, dx$$

SOLUTION: Another improper integral. Observe that $\frac{x}{x^2-4} > \frac{1}{x}$

$$\lim_{b \to \infty} \int_4^b \frac{x}{x^2 - 4}\, dx > \lim_{b \to \infty} \int_4^b \frac{1}{x}\, dx$$

The integral on the right is divergent, hence so is the one on the left.

(e) Discuss

$$\int_4^\infty \frac{x}{(x^2 - 4)^2}\, dx$$

SOLUTION: We expect this one to converge, since it behaves roughly the same as $\frac{1}{x^3}$, whose integral converges on the region $[4, \infty)$. We can prove this one by a direct computation using a u-substitution ($u = x^2 - 4$).

$$\lim_{b \to \infty} \int_4^b \frac{x}{(x^2 - 4)^2}\, dx = \lim_{b \to \infty} \frac{1}{2} \int_{12}^{b^2-4} \frac{du}{u^2}$$

$$= \lim_{b \to \infty} \frac{-1}{b^2 - 4} + \frac{1}{12} = \frac{1}{12}$$

Thus, the integral converges.

3. Consider the area under the curve $y = xe^x$ for $0 \le x \le 1$.

(a) Set up the integral that gives this area.

SOLUTION:

$$A = \int_a^b f(x)\, dx = \int_0^1 xe^x\, dx$$

(b) Set up the integral that gives the volume when this area is revolved around the x-axis.

SOLUTION: Discs:

$$V = \int_a^b \pi(f(x))^2\, dx = \int_0^1 \pi x^2 e^{2x}\, dx$$

(c) Set up the integral that gives the volume when this area is revolved around the y-axis.

SOLUTION: Shells

$$V = \int_a^b 2\pi x f(x)\, dx = \int_0^1 2\pi x^2 e^x\, dx$$

(d) Set up the integral that gives the volume when this area is revolved around the line $x = 1$.

SOLUTION: Shells, this time with a different radius:

$$V = \int_a^b 2\pi(1-x)f(x)\, dx = \int_0^1 2\pi(1-x)xe^x\, dx$$

(e) Set up the integral that gives the volume when this area is revolved around the line $y = -2$.

SOLUTION: Washers:

$$V = \int_a^b \pi(R_o^2 - R_i^2)\, dx = \int_0^1 (2 + xe^x)^2 - 4\, dx$$

4. Determine the following integrals

(a)

$$\int (x^2+1)e^{-x}\, dx$$

SOLUTION: Integration by parts, twice. Let $u = x^2 + 1,\ dv = e^{-x}\, dx$. Then $du = 2x\, dx,\ v = -e^{-x}$, and

$$\int (x^2+1)e^{-x}\, dx = -(x^2+1)(e^{-x}) + \int (2x)e^{-x}\, dx$$

Parts again for the second integral, $u = 2x,\ dv = e^{-x}\, dx$, giving $du = 2\, dx$ and $v = -e^{-x}$.

$$\int (2x)e^{-x}\, dx = -2xe^{-x} + \int 2e^{-x}$$

$$\int (x^2+1)e^{-x}\, dx = -(x^2+1)(e^{-x}) - 2xe^{-x} - 2e^{-x} + C$$

(b)

$$\int \cos^2(x)\tan^3(x)\, dx$$

SOLUTION: Rewrite the integrand as $\cos^2(x)\frac{\sin^3(x)}{\cos^3(x)}$. Thus

$$\int \cos^3(x)\tan^3(x)\, dx = \int \frac{\sin^3(x)}{\cos(x)}\, dx = \int \frac{1-\cos^2(x)}{\cos(x)} sin(x)\, dx,$$

in which we make the substitution $u = \cos x$, giving

$$-\int \frac{1-u^2}{u}\, du = -\int \frac{1}{u} - u\, du = -\ln|u| + \frac{u^2}{2} + C = \frac{\cos^2(x)}{2} - \ln|cos(x)| + C$$

(c)

$$\int \frac{t^5}{\sqrt{t^2+1}}\, dt$$

SOLUTION: Since we have the term $\sqrt{t^2+1}$, we want to make a substitution of the form $t = \tan(\theta)$. Then $dt = \sec^2(\theta)\, d\theta$, and the integral becomes

$$\int \frac{t^5}{\sqrt{t^2+1}}\, dt = \int \frac{tan^5(\theta)}{\sqrt{\tan^2(\theta)+1}} \sec^2(\theta)\, d\theta$$
$$= \int \frac{\tan^5(\theta) sec^2(\theta)}{\sec(\theta)}\, d\theta = \int \tan^5(\theta)\sec(\theta)\, d\theta.$$

We want to lay off a $\sec(\theta)\tan(\theta)$ and use identities to write $\tan^4(\theta) = (sec^2(\theta) - 1)^2$, which, with the substitution $u = \sec(\theta)$ will give

$$\int \tan^5(\theta)\sec(\theta)\, d\theta = \int (sec^2(\theta)-1)^2 \sec(\theta)\tan(\theta)\, d\theta = \int (u^2-1)^2\, du = \frac{u^5}{5} - \frac{2u^3}{3} + u + C$$

Going back and rewriting everything in terms of our original variables gives:

$$\frac{u^5}{5} - \frac{2u^3}{3} + u + C = \frac{\sec^5(\theta)}{5} - \frac{2\sec^3(\theta)}{3} + \sec(\theta) + C = \frac{(t^2+1)^{\frac{5}{2}}}{5} - \frac{2(t^2+1)^{\frac{3}{2}}}{3} + \sqrt{t^2+1} + C$$

whew.

(d)

$$\int \frac{x-6}{x^2+4x+3}\, dx$$

SOLUTION: Straight Partial Fractions. The denominator factors as $(x+1)(x+3)$, and so

$$\int \frac{x-6}{x^2+4x+3}\, dx = \int \frac{A}{x+1} + \frac{B}{x+3}\, dx$$

Crossmultiplying terms will give $A = \frac{-7}{2}$ and $B = \frac{9}{2}$, so

$$\int \frac{x-6}{x^2+4x+3}\, dx = \frac{-7}{2}\ln|x+1| + \frac{9}{2}\ln|x+3| + C$$

(e)

$$\int \frac{\sqrt{x-4}}{x}\,dx$$

SOLUTION: This one requires a rationalizing substitution. Let $u = \sqrt{x-4}$, then $x = u^2 + 4$ and $dx = 2u\,du$. Then

$$\int \frac{\sqrt{x-4}}{x}\,dx = \int \frac{2u^2}{u^2+4}\,du = 2\int \frac{u^2+4-4}{u^2+4}\,du$$

$$= 2\int 1 - \frac{4}{u^2+4} + C\,du = 2(u - 2\arctan(\frac{u}{2})) = 2\sqrt{x-4} - 4\arctan(\frac{\sqrt{x-4}}{2}) + C$$

(f)

$$\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx$$

SOLUTION: This one is an integration by parts with a u substitution thrown in. First, substitute $u = \sqrt{x}$ and $du = \frac{1}{2sqrtx}\,dx$. Then

$$\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx = \frac{1}{2}\int \arctan u\,du$$

We can integrate by parts to find the antiderivative of $arctan(x)$, or we can recall it from the table of integrals.

$$\frac{1}{2}\int \arctan u\,du = \frac{1}{2}(u\arctan u - \frac{1}{2}\ln u^2 + 1) + C$$

$$= \frac{1}{2}\sqrt{x}\arctan\sqrt{x} - \frac{1}{4}\ln|x| + 1 + C$$

5. Set up the integral to compute the length of one period of the curve $y = \sin x$. Also, set up the integral to compute the surface area of the solid generated by revolving this curve about the x-axis.

SOLUTION:

$$ArcLength = \int_a^b \sqrt{1 + (f'(x))^2}\,dx = \int_0^\pi \sqrt{1 + cos^2(x)}\,dx$$

$$SurfaceArea = \int_a^b 2\pi x\sqrt{1 + (f'(x))^2}\,dx = \int_0^\pi 2\pi x\sqrt{1 + cos^2(x)}\,dx$$

6. Define a sequence $\{a_n\}_{n=1}^{\infty}$ by $a_1 = 1$ and $a_n = a_{n-1}^2 - 1$. What are the first six terms of the sequence? Does the sequence approach a limit? If so what? If we define $b_n = a_n^n$, does the series $\sum_{n=1}^{\infty} b_n$ converge?

SOLUTION:

$$a_1 = 1, a_2 = 0, a_3 = -1, a_4 = 0, a_5 = -1, a_6 = 0$$

The sequence does not approach a limit. If we raise the nth term to the nth power, then the series does not converge. This relation is very dependent on the initial condition.

7. Determine the convergence or divergence of the following series.

(a)

$$\sum_{n=1}^{\infty} \frac{n^2 - 1}{2 - n^3}$$

SOLUTION: This series behaves like $\sum \frac{1}{n}$, so, by the limit comparison test, the series diverges.

(b)

$$\sum_{n=1}^{\infty} \ln\left(\frac{2n}{n-3}\right)$$

SOLUTION: (Note: Apologies on the limits. Treat the problem as though they made sense) As n gets large, $\frac{2n}{n-3}$ approaches 2. Thus, the terms in the series are approaching $\ln 2$. Since the terms do not approach zero, the series automatically Diverges.

(c)

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$$

SOLUTION: Using the root test, we see that the nth root of a_n is $\frac{n^2}{1+2n^2}$. As n gets large, the root approaches $\frac{1}{2}$, which is less than 1, thus the series converges absolutely.

(d)

$$\sum_{n=1}^{\infty} \frac{(-1)^n n + 1}{2n^2 + 1}$$

SOLUTION: The series, after splitting the numerator, becomes two series. The second of these series, $\sum \frac{1}{2n^2+1}$ converges absolutely by the p-series test. The first of these, $\sum \frac{(-1)^n n}{2n^2+1}$ behaves like $\sum \frac{(-1)^n}{n}$, which converges conditionally. Hence, the whole series converges conditionally.

(e)

$$\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$$

SOLUTION: This one will use the ratio test. Taking

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \frac{n+1}{(2n+2)(2n+1)} = e \cdot 0 = 0$$

Thus, the series converges.

(f)

$$\sum_{n=1}^{\infty} \frac{3^n}{4^n + 5^n}$$

SOLUTION: The 5^n term will dominate the denominator, hence this series behaves like $\sum \left(\frac{3}{5}\right)^n$, which is geometric, with ratio less than one, thus it converges. Note that in doing the limit comparison test, life is much easier if you set $\left(\frac{3}{5}\right)^n = a_n$ and $\frac{3^n}{4^n+5^n} = b_n$

8. Determine

$$\int \frac{e^x}{x}\, dx \text{ and } \int \frac{e^{-x}}{x}\, dx$$

by using series.

SOLUTION: Recall that the series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Hence, the first integral will be found by integrating $\sum_{n=0}^{\infty} \frac{x^{n-1}}{n!}$, and so the integral will be $\ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$, or

$$\int \frac{e^x}{x}\, dx = \ln|x| + x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \dots$$

.

To get the corresponding series for e^{-x}, one can just replace x with $-x$ in the first solution.

9. Determine c so that

$$f(x) = \begin{cases} \frac{c}{x^2} & x > 2 \\ 0 & x < 2 \end{cases}$$

is a probability density function.

SOLUTION: For $f(x)$ to be a probability distribution function, its integral over the entire real line must be 1. Thus

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_2^{\infty} \frac{c}{x^2}\, dx = \lim_{b\to infty} \frac{-c}{x}|_2^b = \frac{c}{2}.$$

Therefore, $c = 2$.

10. Find the center of mass of a plate in the shape of the area under the curve $y = \sin 2x$ of density ρ, between $x = 0$ and $x = \frac{\pi}{2}$.

SOLUTION: The plate is symmetric about the line $x = \frac{\pi}{4}$, so $\overline{x} = \frac{\pi}{4}$. We need to calculate the moment about the x-axis, and divide by the mass in order to get $\overline{y}$.

$$\overline{y} = \frac{M_x}{m} = \frac{\frac{\rho}{2}\int_0^{\frac{\pi}{2}} \sin^2(2x)\, dx}{\rho\int_0^{\frac{\pi}{2}} \sin(2x)\, dx} = \frac{\pi}{8}$$

We expect this to be a bit less than $\frac{1}{2}$ due to the nature of the shape of the plate, and it is.

11. Snow is falling on the ground at the rate of 4 inches/minute. It is melting at a rate of 75% How much snow is on the ground after 5 hours? How much snow remains on the ground if it continues to snow indefinitely?

    SOLUTION: Let $S(t)$ be the amount of snow on the ground at time t. Then S is modeled by

    $$\frac{dS}{dt} = 4 - .75S$$

    Solving for S gives $S = Be^{-.75t} + \frac{16}{3}$, with $B = -\frac{16}{3}$ when $t = 0$. When $t = 5$, there is 5.208 inches of snow on the ground. Long term, there is $\frac{16}{3}$ inches of snow on the ground.