KEY

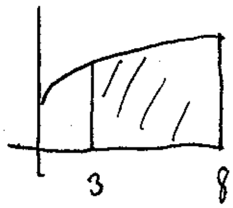
Math 126: Exam the First

October 19, 2012

You have 90 minutes to complete this closed-book, closed-notes, closed colleague quiz. You may use a calculator for arithmetic, trigonometric, and exponential/logarithmic functions only (ie, NO PLOTTING and NO CALCULUS). You are encouraged to read through the exam before beginning work.

1. Let $f(x) = \sqrt{x+1}$.

   (a) Estimate the area bound by $y = f(x)$, $y = 0$, $x = 3$ and $x = 8$ by using 5 subdivisions. Give both an upper and lower estimate of the area.



x	3	4	5	6	7	8
f(x)	2	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	$\sqrt{8}$	3

LE $= 2\cdot 1 + \sqrt{5}\cdot 1 + \sqrt{6}\cdot 1 + \sqrt{7}\cdot 1 + \sqrt{8}\cdot 1$

UE $= \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8} + 3$

   (b) Find the exact area using calculus.

$$\int_3^8 \sqrt{x+1}\,dx = \frac{2}{3}(x+1)^{3/2}\Big|_3^8 = \frac{2}{3}\left[9^{3/2} - 4^{3/2}\right] = \frac{2}{3}[27-8] = \frac{38}{3}$$

   (c) Revolve the area about $y = -1$ and find the volume generated.

(WASHERS

$$V = \int_a^b \pi\left(r_{out}^2 - r_{in}^2\right)dx$$

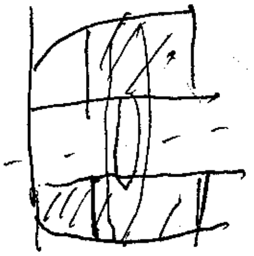
$$= \int_a^b \left(\sqrt{x+1}+1\right)^2 - (1)^2\,dx$$

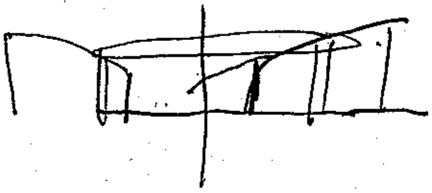
$$= \int_3^8 x+1+2\sqrt{x+1}\,dx$$

$$= \left[\frac{x^2}{2} + x + \frac{4}{3}(x+1)^{3/2}\right]_3^8$$

$$= \frac{64}{2} + 8 + 36 - \left[\frac{9}{2} + 3 + \frac{32}{3}\right]$$

$$= 347/4$$





$$V_{shell} = 2\pi rh\,dr$$

$$= 2\pi xy\,dx$$

$$\int_3^8 2\pi x\sqrt{x+1}\,dx \longrightarrow \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\int (x+1)^{3/2}\,dx$$

IBP! $u = x$ $dv = \sqrt{x+1}$

$du = dx$ $v = \frac{2}{3}(x+1)^{3/2}$

$$= \frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2}\Big]_3^8$$

$$= \left(\frac{2}{3}\cdot 8\cdot 27\right) - \frac{4}{15}\cdot(243)\Big) - \left(\frac{2}{3}\cdot 3\cdot 8\right) + \frac{4}{15}\cdot 32$$

$$= 128 - 56.2\overline{66} = 71.7\overline{33}\cdot 2\pi$$

2. Find each of the following antiderivatives.

(a) $\int t^2 e^{t^3}\, dt \quad = \quad \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{t^3} + C$

let $u = t^3$

$du = 3t^2 dt$

(b) $\int \frac{x^2+1}{x^2-4}\, dx \quad = \int 1 + \frac{5}{x^2-4} dx = \int 1 - \frac{5/4}{x+2} + \frac{5/4}{x-2} dx$

$= x - \frac{5}{4}\ln|x+2| + \frac{5}{4}\ln|x-2| + C$

Division or manipulation

$$\begin{array}{r} 1 + \frac{5}{x^2-4} \\ y^2-4 \overline{\smash{)}\, x^2 + 1} \\ \underline{y^2 - 4} \\ +5 \end{array}$$

$\frac{5}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$

$Ax - 2A + Bx + 2B = 5$

$A + B = 0$

$\underline{-2A + 2B = 5}$

$\rightarrow 4B = 5$

$B = \frac{5}{4}$; $A = -5/4$

sec power is even!

(c) $\int \sec^4(x)\tan^4(x)\,dx$

$= \int \sec^2 x \;\tan^4 x\; \sec^2 x\, dx$

$= \int (\tan^2 x + 1) \tan^4 x\; \sec^2 x\, dx$ $\qquad u = \tan x$ $\quad du = \sec^2 x$

$\int (u^2+1)\, u^4\, du$

$= \int u^6 + u^4\, du$

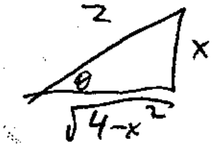
$= \frac{u^7}{7} + \frac{u^5}{5} + C = \frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + C$

(d) $\int \sqrt{4-x^2}\,dx$ $= \int \sqrt{4 - 4\sin^2\theta}\cdot 2\cos\theta\, d\theta$

let $x = 2\sin\theta$

then $dx = 2\cos\theta\, d\theta$

$= \int 4\cos^2\theta\, d\theta$



$= \int 4\left(\frac{1+\cos 2\theta}{2}\right) d\theta$

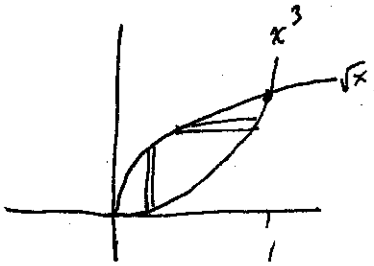
$= \int 2 + 2\cos 2\theta\; d\theta$

$= 2\theta + \sin 2\theta$

$= 2\theta + 2\sin\theta\cos\theta$

$= 2\arcsin\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$

3. The area in the first quadrant bound by $f(x) = x^3$ and $g(x) = \sqrt{x}$ can be found using horizontal or vertical strips. Solve the integral both ways and check that your answers are the same.



V strips : $\int_0^1 \sqrt{x} - x^3\, dx = \frac{2}{3}x^{3/2} - \frac{x^4}{4}\Big|_0^1$

$= \frac{2}{3} - \frac{1}{4} = \boxed{5/12}$

H strips : $\int_0^1 y^{1/3} - y^2\, dx = \frac{3}{4}y^{4/3} - \frac{y^3}{3}\Big|_0^1$

$= \frac{3}{4} - \frac{1}{3} = \boxed{\frac{5}{12}}$

4. Consider

$$\int \sin(x)\cos(x)\,dx.$$

We can solve this via a u-substitution, substituting either $u = \cos(x)$ or $u = \sin(x)$. The two substitutions seem to give different answers. Do they? Explain.

$\int \sin x \cos x \, dx$

$u = \cos x$

$du = -\sin x\, dy$

$= -\int u\, du$

$= -\frac{u^2}{2}$

$= \frac{-\cos^2 x}{2} + C$

or

$\int \sin x \cos x \, dx$

$u = \sin x$

$du = \cos x\, dx$

$\int u\, du$

$= \frac{u^2}{2}$

$= \frac{\sin^2 x}{2} + C$

which appear to be different, but with the appropriate choices of C, they only differ by a constant!

$-\frac{\cos^2 x}{2} = -\frac{(1-\sin^2 x)}{2} = \frac{\sin^2 x - 1}{2} = \frac{\sin^2 x}{2} - \frac{1}{2}$

difference is a <u>constant</u>

Do ONE of the next TWO problems

5. A conical party hat of height 6 inches and radius 2 inches is filled with (root) beer to a depth of 4 inches. SET UP but DON'T COMPUTE the integral for the amount of work done in pumping the (root) beer to the top of the cone. The weight of the (root) beer is .04 lb/in^3.

6. SET UP but DON'T COMPUTE the integrals to find the centroid of the area in Problem 1.

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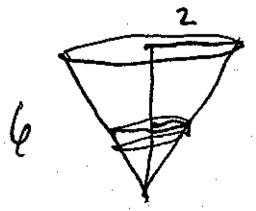
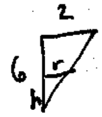


$V_{DSK} = \pi r^2 h = \pi r^2 \, dh = \pi \left(\frac{1}{3}h\right)^2 dh = \frac{\pi h^2}{9}$

$F_{DSK} = \frac{\pi h^2}{9}(.04)$

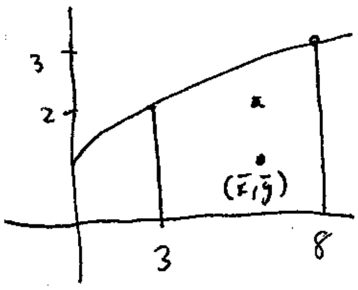


Work dsk $= \frac{\pi h^2}{9}(.04)(6-h)$

$\frac{r}{h} = \frac{2}{6}$ so $r = \frac{1}{3}h$

Total work $= \int_0^4 .04 \frac{\pi h^2}{9}(6-h)\,dh = \frac{.04\pi}{9}\int_0^4 h^2(6-h)\,dh$

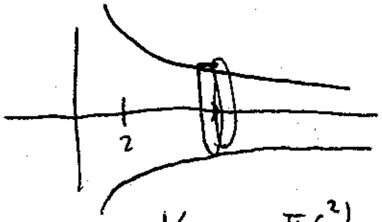
6]



$\bar{x} = \frac{\int x \cdot \text{area}(\text{strip})}{\int \text{area}(\text{strip})} = \frac{\int_3^8 x\sqrt{x+1}\,dx}{\int_3^8 \sqrt{x+1}\,dx}$

$\bar{y} = \frac{\int y \cdot \text{area Strip}}{\int \text{area}(\text{Strip})} = \frac{\int_0^2 y(8-3)\,dy + \int_2^3 y\cdot(8-(y^2-1))\,dy}{\int_0^2 (8-3)\,dy + \int_2^3 8-(y^2-1)\,dy}$

7. In class we showed that the area under $f(x) = 1/x$ (and above the x-axis) to the right of $x = 2$ can be set up as a divergent improper integral. Suppose we revolve that area around the x-axis. Is the volume finite or infinite? Explain.



$$V_{Disk} = \pi r^2 h$$
$$= \pi\left(\frac{1}{x}\right)^2 dx$$

$$\int_2^\infty \frac{\pi}{x^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{\pi\,dx}{x^2}$$
$$= \lim_{b\to\infty} \pi\left[-x^{-1}\right]_2^b$$
$$\lim_{b\to\infty} \pi\left[\frac{-1}{b} + \frac{1}{2}\right] = \frac{\pi}{2}$$

So even though the area is "infinite", the volume generated is finite

8. Suppose that a car moves with velocity $v(t) = t^2 + 5t$. Find the average velocity of the car between $t = 3$ and $t = 7$.

$$v(t) = t^2 + 5t$$

average velocity $= \dfrac{1}{7-3}\displaystyle\int_3^7 t^2 + 5t\,dt$

$$= \frac{1}{4}\left[\frac{t^3}{3} + \frac{5t^2}{2}\right]_3^7$$
$$= \frac{1}{4}\left[\frac{7^3}{3} + \frac{5\cdot 7^2}{2} - \frac{3^3}{3} - \frac{5\cdot 3^2}{2}\right] =$$
$$\frac{1}{4}\left[236\tfrac{5}{6} - \left(31\tfrac{1}{2}\right)\right] = \frac{1}{4}\left[205\tfrac{1}{3}\right]$$
$$= 51\tfrac{1}{3}$$