m 125 f 12 prac Exam Key

1. a) By the Fundamental Theorem of Calculus

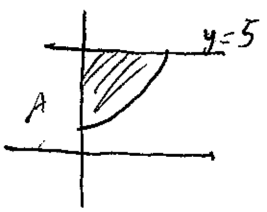
$$f(x) = \int_2^x \cot(t^3) + 2t \, dt = F(x) - F(2) \quad \text{where } F'(t) = \cot(t^3) + 2t.$$

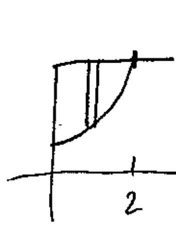
Thus, $f'(x) = F'(x) - \underset{\substack{\uparrow \\ F(2) \text{ is a constant}}}{0} = \cot(x^3) + 2x.$

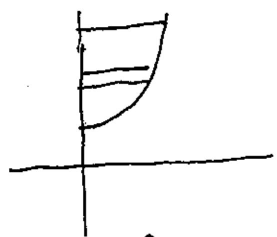
b) Here, $f(x) = \int_2^{x^2} \cot(t^3) + 2t \, dt = F(x^2) - F(2)$ where $F'(t) = \cot(t^3) + 2t$

So $f'(x) = \underset{\text{chain rule}}{2x} F'(x) = 2x(\cot(x^6) + 2x^2)$

2 a)





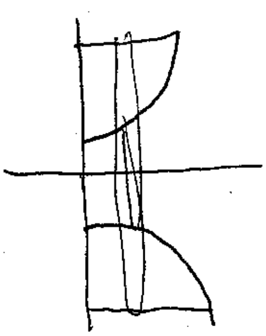


$$\cancel{A = \int_0^2 x^2 + 1 \, dx = \frac{x^3}{3} + x \Big|_0^2 = \frac{8}{3} + 2 = \frac{14}{3}}$$

$$\int_1^5 \sqrt{y-1} \, dy = \frac{2}{3}(y-1)^{3/2} \Big|_1^5 = \frac{2}{3}(5-1)^{3/2} - \frac{2}{3}(1-1)^{3/2} = \frac{2}{3}(8) = \frac{16}{3}$$

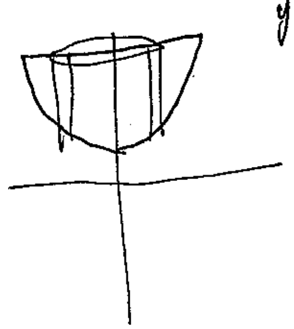
$$\int_0^2 5 - (x^2+1) \, dx = \int_0^2 4 - x^2 \, dx = 4x - \frac{x^3}{3} \Big|_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

2b.



x-axis:

$$\pi \int_0^2 5^2 - (x^2+1)^2 \, dx$$



y-axis

$$\int_0^2 2\pi x \, (5 - (1+x^2)) \, dx$$

3a. $\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx$ let $u=\sqrt{x}$ then $du = \frac{1}{\sqrt{x}}\cdot\frac{1}{2}\,dx$ $= \int 2 \arctan u \, du$

Dry IBP

$w = \arctan u \qquad dv = du$

$dw = \frac{1}{1+u^2}\,du \qquad v = u$

$2\int \arctan u \, du = 2u\arctan u - 2\int \frac{u}{1+u^2}\,du$ ✓subst

$= 2u\arctan u - 2\ln|1+u^2| + C$

$= \underline{2\sqrt{x}\arctan\sqrt{x} - 2\ln|1+x| + C}$

b. $\int x^5 \cos x^3 \, dx$ IBP $= \frac{x^3}{3}\sin x^3 - \int x^2 \sin x^3 \, dx$ ✓subst

$u = x^3 \qquad dv = x^2\cos x^3$

$du = 3x^2\,dx \qquad v = \frac{1}{3}\sin x^3$

$= \underline{\frac{x^3}{3}\sin x^3 + \frac{1}{3}\cos x^3 + C}$

c. $\int (x+1)\cos(x^2+2x)\,dx = \frac{1}{2}\int \cos u \, du$

$u = x^2+2x$

$du = 2x+2\,dx$

$= \frac{1}{2}\sin u + C = \underline{\frac{1}{2}\sin(x^2+2x) + C}$

d. $\int \frac{2x}{x^2+3x-10}\,dx = \int \frac{2x}{(x+5)(x-2)}\,dx = \int \frac{A}{x+5} + \frac{B}{x-2}\,dx$

$A(x-2) + B(x+5) = 2x \rightarrow \begin{array}{r} A+B=2 \\ -2A+5B=0 \\ \hline 7B=4 \end{array}$

$(A+B)x - 2A + 5B = 2x$

$B = 4/7 \qquad A = 10/7$

$\int \frac{10/7}{x+5} + \frac{4/7}{x-2}\,dx = \underline{\frac{10}{7}\ln|x+5| + \frac{4}{7}\ln|x-2| + C}$

e) $\int \frac{t^2-1}{t^2+1}\,dt = \int \frac{t^2+1-2}{t^2+1}\,dt = \int \frac{t^2+1}{t^2+1} - \frac{2}{t^2+1}\,dt = \int 1 - \frac{2}{t^2+1}\,dt = \underline{t - 2\arctan t + C}$

f) $\int \sec^4(x) \tan^4(x)\,dx$ Power of sec x is even...

$\int \sec^2 x \tan^4 x \sec^2 x\,dx$

$u = \tan x$

$\int (1+\tan^2 x)\tan^4 x \sec^2 x\,dx$ $du = \sec^2 x\,dx$

$\int (1+u^2)u^4\,du = \int u^4 + u^6\,du = \frac{u^5}{5} + \frac{u^7}{7} + C = \underline{\frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C}$

4a) $\int \cos^2(2x)\sin(2x)\,dx = -\frac{1}{2}\int u^2\,du = -\frac{1}{2}\frac{u^3}{3} + C = \underline{-\frac{1}{6}\cos^3(2x) + C}$

$u = \cos 2x$

$du = -2\sin 2x\,dx$

b) $\int \ln(x^2)\,dx \xrightarrow[\text{Dry IBP}]{\text{method I}}$ $u = \ln x^2$, $du = \frac{1}{x^2}\cdot 2x\,dx$ $dv = dx$, $v = x$ $\mapsto \int \ln x^2\,dx = x\ln x^2 - \int \frac{2x}{x}\,dx$

$= \underline{x\ln x^2 - 2x + C}$

method 2 $\downarrow$ Rules of logs

equiv by rules of logs.

$\int 2\ln x\,dx = 2(x\ln x - x) + C = \underline{2x\ln x - 2x + C}$

c) $\int \frac{5}{x^2+x-6}\,dx = \int \frac{5}{(x+3)(x-2)}\,dx = \int \frac{A}{x+3} + \frac{B}{x-2}\,dx$

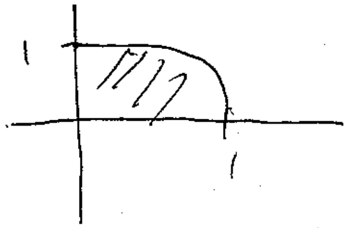
$A(x-2) + B(x+3) = 5 \rightarrow$ $A + B = 0$, $-2A + 3B = 5$

$(A+B)x - 2A + 3B = 5$ $A = -1$ $B = 1$.

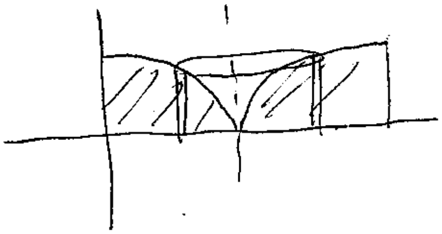
$\int \frac{-1}{x+3} + \frac{1}{x-2}\,dx = \underline{-\ln|x+3| + \ln|x-2| + C}$

(4)

5. $f(x) = 1-x \quad y = \sqrt{1-x} \quad x = 1-y^2$

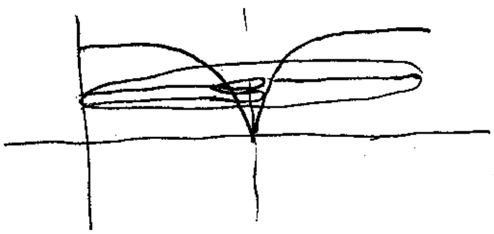
a) 

$A = \int_0^1 \sqrt{1-x}\,dx = -\frac{2}{3}(1-x)^{3/2}\Big|_0^1 = -\frac{2}{3}\left((1-1)^{2/3} - (1-0)^{2/3}\right)$
$= 2/3.$

b. i 

$V_{shells} = \int_0^1 2\pi \underset{\uparrow\ \text{radius}}{(1-x)} \underset{\uparrow\ \text{height}}{\sqrt{1-x}}\ \underset{\uparrow\ \text{thickness}}{dx}$

$= \int_0^1 2\pi(1-x)^{3/2}\,dx = -2\pi\left(\frac{2}{5}\right)(1-x)^{5/2}\Big|_0^1$
$= -\frac{4\pi}{5}(0-1) = \frac{4\pi}{5}.$

ii. 

$V_{washers} =$
$\int_0^1 \pi \left(\underset{\uparrow\ \text{rout}}{1}\right)^2 - \left(\underset{\uparrow\ \text{rin}}{1-x}\right)^2 dy$

$= \pi \int_0^1 1 - y^4\,dy$
$= \pi \left[y - \frac{y^5}{5}\right]_0^1 = \pi\left[\frac{4}{5}\right] = \frac{4\pi}{5}.$

6. $\int_2^\infty \frac{dx}{\sqrt{x}(1+x)}$ We expect this to converge, since its like $\int_2^\infty \frac{1}{x^{3/2}}\,dx$
with $p = 3/2 > 1$.

Subst.
let $u = \sqrt{x}$
or $x = u^2$
then $dx = 2u$

$\lim_{b\to\infty} \int_{x=2}^{x=b} \frac{2u}{u(1+u^2)}\,du$

$= \lim_{b\to\infty} \int_{x=2}^{x=b} \frac{2}{1+u^2}\,du = \lim 2\arctan u \Big]_{x=2}^{x=b} = \lim_{b\to\infty} 2\arctan\sqrt{x}\Big|_2^b$

$= \lim_{b\to\infty} 2\arctan\sqrt{b} - 2\arctan\sqrt{2}$
$= \pi - 2\arctan\sqrt{2} \to 80^\circ$

Convergent

7.

a) $\int_3^{10} \frac{x}{x^2-4}\,dx = \int_3^{10} \frac{A}{x+2} + \frac{B}{x-2}\,dx = \int_3^{10} \frac{1}{2}\left(\frac{1}{x+2} + \frac{1}{x-2}\right)dx$

$A(x-2) + B(x+2) = x$

$(A+B)x - 2A + 2B = x$

$A + B = 1$

$-2A + 2B = 0 \quad A = B = \frac{1}{2}$

$= \frac{1}{2} \ln|x+2| + \ln|x-2| \Big|_3^{10}$

$= \frac{1}{2}\left(\ln 12 + \ln 8 - \ln 5 - \ln 1\right)$

$\underset{\text{rules of logs}}{=} \frac{1}{2}\ln\left(\frac{96}{5}\right)$.

b) $\int_{-1}^{1} \frac{x}{x^2-4}\,dx$ We save ourselves the hassle of integration by recognizing that $f(x) = \frac{x}{x^2-4}$ is an odd function so $\int_{-a}^{a} f(x) = 0$.

c) $\int_0^5 \frac{x}{x^2-4}$ There is a break in the domain, with the asymptote @ $x = 2$.

Improper Integral

Thus, we'd need to compute $\lim_{b\to 2^-} \int_0^{b} \frac{x}{x^2-4}\,dx + \lim_{b\to 2^+} \int_b^5 \frac{x}{x^2-4}\,dx$

each of which diverge in area.

d) $\int_4^{\infty} \frac{x}{x^2-4}\,dx$ again, an improper integral.

$\lim_{b\to\infty} \frac{1}{2}\left[\ln|x+2| + \ln|x-2|\right]_4^b \to \infty$. So the integral is Divergent.

e) $\int_4^{\infty} \frac{x}{(x^2-4)^2}\,dx$ Here, we expect the integral to converge, since the effective power of x is $-3 < -1$.

$\downarrow$ subst

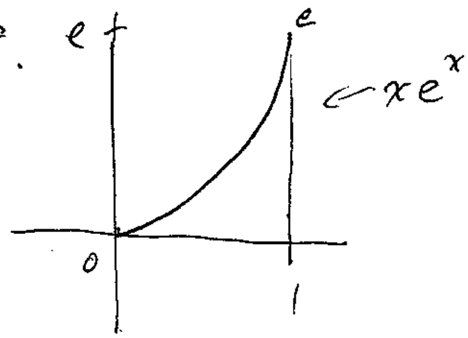
$u = x^2 - 4$

$du = 2x\,dx$

$\lim_{b\to\infty} \int_{x=4}^{x=b} \frac{1}{2}\frac{du}{u^2} = \frac{-1}{2u}\Big|_{x=4}^{x=b} = \lim_{b\to\infty} -\frac{1}{2(x^2-4)}\Big|_4^b$

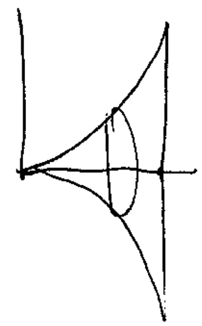
$= \lim_{b\to\infty} -\frac{1}{2(b^2-4)} + \frac{1}{2\cdot 12} = 1/24$

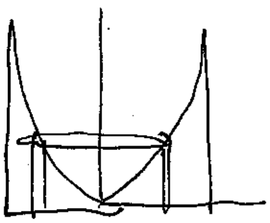
8.



a) $\int_0^1 xe^x\,dx$

b) $\int_0^1 \pi (xe^x)^2\,dx$ Disks.

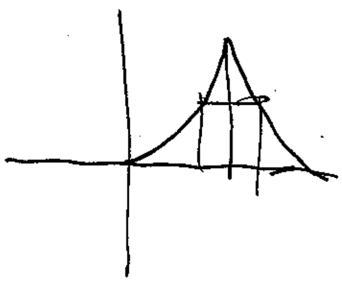


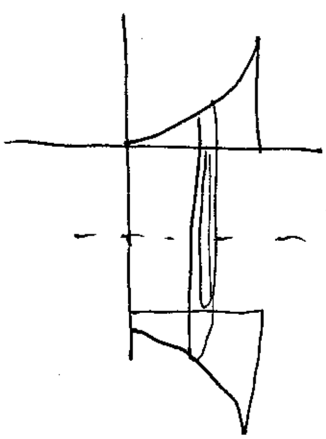


c) $\int_0^1 2\pi x(xe^x)\,dx$

d) $\int_0^1 2\pi (1-x)\, xe^x\,dx$.

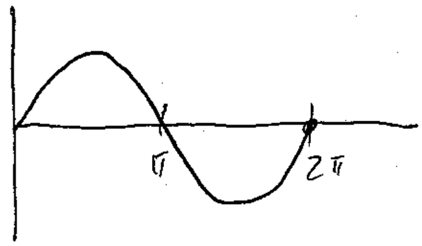
shells





e) $\int_0^1 \pi (2+xe^x)^2 - 2^2\,dx$ washers

9.



$$\text{Arc length} = \int_0^b \underbrace{\sqrt{1+(f'(x))^2}}_{ds}\, dx$$

$$= \int_0^{2\pi} \sqrt{1+(\cos^2 x)}\, dx$$

$f(x) = \sin x$

$f'(x) = \cos x$

$$\text{Surface area} = \int_a^b 2\pi\, r \underbrace{\sqrt{1+f'(x)^2}}_{ds}\, dx$$

$$= \int_0^{2\pi} 2\pi \sin x \sqrt{1+\cos^2 x}\, dx.$$

10. If $f(x) \begin{cases} c/x^2 & x > 2 \\ 0 & x < 2 \end{cases}$ is a p.d.f.

Then $\int_{-\infty}^{\infty} f(x)\,dx = 1$ so $\int_2^{\infty} \frac{c}{x^2}\,dx = 1$

$$\lim_{b\to\infty} -\frac{c}{x}\Big|_2^b = 1,\ \lim_{b\to\infty} -\cancel{\frac{c}{b}}^{\,0} + \frac{c}{2} = 1$$

$\underline{\text{so } c = 2.}$

11. $\frac{dS}{dt} = 4 - .75S = -.75\left(S - \frac{16}{3}\right)$

(sorry for the implausible units!)

$$\frac{dS}{S - 16/3} = -.75\,dt \rightarrow \ln|S - 16/3| = -.75t + C$$

$$S - \frac{16}{3} = Be^{-.75t}$$

$$S = \frac{16}{3} + Be^{-.75t}$$

Assuming $S(0) = 0$.

(would be given on the test)

$$\longrightarrow S = \frac{16}{3} - \frac{16}{3}\cancel{B}e^{-.75t}$$

$$S(5) = \frac{16}{3}\left(1 - e^{-\frac{15}{4}}\right)$$

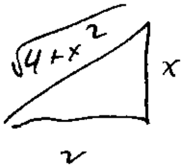
$$S_{\to\infty} = 16/3 \text{ inches.}$$

12. $\int_2^\infty \frac{1}{\sqrt{4+x^2}}\,dx$ we expect this to diverge since it's effectively $\int_a^\infty \frac{1}{x}\,dx$.

↓ trig sub

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$



$$\lim_{b\to\infty}\int_{x=2}^{x=b} \frac{2\sec^2\theta\,d\theta}{\sqrt{4+4\tan^2\theta}} = \lim_{b\to\infty}\int_{x=2}^{x=b}\frac{2\sec^2\theta\,d\theta}{2\sec\theta}$$

$$= \lim_{b\to\infty}\int_{x=2}^{x=b}\sec\theta\,d\theta = \lim_{b\to\infty}\left[\ln|\sec\theta+\tan\theta|\right]_{x=2}^{x=b}$$

$$= \lim_{b\to\infty}\left[\ln\left|\frac{\sqrt{4+x^2}}{2}+\frac{x}{2}\right|\right]_2^b$$

$$= \lim_{b\to\infty}\underbrace{\ln\left(\frac{\sqrt{4+b^2}+b}{2}\right)}_{\to\,\infty} - \ln\left|\frac{\sqrt{8}}{2}+1\right|$$

So the integral diverges.

13. $\frac{dH}{dt} = -.2\,(H-20)$

$$\frac{dH}{H-20} = -.2\,dt \rightarrow \ln|H-20| = -.2t + C$$

$$\rightarrow H-20 = Be^{-.2t}$$

$$\rightarrow H = 20 + Be^{-.2t}$$

$H(0)=90 \rightarrow$

$90 = 20 + Be^{0} \Rightarrow 70 = B$

$(70 = \Delta_{init} T)$

$$H = 20 + 70e^{-.2t}$$

When does $H = 50$? $50 = 20 + 70e^{-.2t}$

$\frac{30}{70} = e^{-.2t}$; $t = \frac{\ln 3/7}{-.2} = 4.24$ minutes

14.

a) $f(x)=\frac{8}{(x+2)^3}$ if

$$\int_0^\infty \frac{8}{(x+2)^3} = 1 \quad ; \quad -\frac{8}{2}(x+2)^{-2}\Big|_0^b = -4(b+2)^{-2} + 4(0+2)^{-2}$$

0 as $b\to\infty$

$$= 1 \text{ as } b\to\infty.$$

So f is a p.d.f.

b) $$P(X\le 1) = \int_0^1 \frac{8}{(x+2)^3}\,dx = -4(x+2)^{-2}\Big|_0^1$$

$$= \frac{-4}{9} + \frac{+4}{4} = 5/9.$$

15). $a_1=1$ $\quad a_n = a_{n-1}^2 - 1$

$a_2 = 1^2 - 1 = 0$

$a_3 = 0^2 - 1 = -1$

$a_4 = (-1)^2 - 1 = 0$

$a_5 = 0^2 - 1 = -1$

$a_6 = (-1)^2 - 1 = 0$

$a_n = 1, 0, -1, 0, -1, 0, -1, 0, -1, 0, \dots$

a_n doesn't approach a limit, as it oscillates.

$b_n = 1, 0, \frac{-1}{3!}, 0, \frac{-1}{5!}, 0, \frac{-1}{7!}, \dots$

$= 1 + \sum_{n=1}^{\infty} \frac{-1}{(2n+1)!}$, which converges by the ratio test

$$1 - \left(\left(\frac{e^1+e^{-1}}{2}\right) - 1\right) = 2 + -\frac{e}{2} + -\frac{e^{-1}}{2}.$$

beyond what we've done in this class.

(16) a) $\sum_{n=1}^{\infty} \frac{n^2-1}{2-n^3} \sim \sum_{n=1}^{\infty} \frac{-1}{n}$

lim t comparsa test $\frac{a_n}{b_n} = \frac{n^2-1}{2-n^3} \cdot \frac{-n}{1} = \frac{n^3-n}{2-n^3} \to +1$ a nonzeofinite cnst.

so both series behave the same, thus $\sum_{n=1}^{\infty} \frac{n^2-1}{2-n^3}$ Diverges

b) $\sum_{n=1}^{\infty} \ln\left(\frac{2n}{n-3}\right)$ as $n \to \infty$, $\ln\left(\frac{2n}{n-3}\right) \to \ln 2 \neq 0$

so the series automaticaly Diverges

c) $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$ Root test: $\lim_{n\to\infty} \sqrt[n]{|a_n|}$

$= \lim_{n\to\infty} \sqrt[n]{\frac{n^{2n}}{(1+2n^2)^n}} = \lim_{n\to\infty} \frac{n^2}{1+2n^2} = 1/2 < 1$

So the series is absolutely convergent

d) $\sum_{n=1}^{\infty} \frac{(-1)^n n+1}{2n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n n}{2n^2+1} + \sum_{n=1}^{\infty} \frac{1}{2n^2+1}$ ⇒ Series is convergent (but only conditionally!)

Converges (Alternating w/ terms decreasing.) Converges (p-series)

e) $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$ Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

→ Convergent!

$= \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}}{(2(n+1))!} \cdot \frac{2n!}{n^n}\right|$

$= \lim_{n\to\infty} \left|\left(\frac{n+1}{n}\right)^n \cdot n+1 \cdot \frac{1}{(2n+1)(2n+2)}\right|$ $\to 0$

$\left(\frac{n+1}{n}\right)^n \to e$, $(2n+1)(2n+2) \sim 4n^2$

f) $\sum_{n=1}^{\infty} \frac{3^n}{4^n+5^n} < \sum_{n=1}^{\infty} \frac{3^n}{4^n+4^n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{3^n}{4^n}$ (10)

geometric, w/ ratio $\frac{3}{4} < 1$, so Convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{3^n}{4^n+5^n}$ is also convergent

17] $\sum_{n=0}^{\infty} \frac{n}{n^2+1}(1-x)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{(n+1)^2+1}(1-x)^{\cancel{n+1}} \cdot \frac{n^2+1}{n\cancel{(1-x)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\underbrace{\frac{n+1}{n}}_{1} \cdot \underbrace{\frac{n^2+1}{(n+1)^2+1}}_{1} \cdot (1-x)\right| = 1-x$$

$|1-x| < 1$

$-1 < 1-x < 1$

$-2 < -x < 0$

$2 > x > 0$

If $x=2$ $\sum_{n=0}^{\infty} \frac{n}{n^2+1}(1-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+1}$ alt, w/ terms decreasing, Convergent

If $x=0$ $\sum_{n=0}^{\infty} \frac{n}{n^2+1}(1-0)^n = \sum_{n=0}^{\infty} \frac{n}{n^2+1} \sim \sum_{n=1}^{\infty} \frac{1}{n}$, so Divergent

Interval of Convergence: $x \in (0, 2]$

(18) (11)

a) $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}} < \sum_{n=1}^{\infty} \frac{n^{1/3}}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{7/6}}$

which is a p-series, $p > 1$, so the series is Convergent, as is $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$

b) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ $\quad \lim_{n\to\infty} \left| \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{n+1}{10} \right| \to \infty$

RATIO

So $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ is Divergent

c) $\sum_{n=1}^{\infty} \frac{\cos(\frac{1}{n})}{n} \sim \sum_{n=1}^{\infty} \frac{1}{n}$

Since $\lim_{n\to\infty} \frac{\frac{\cos(\frac{1}{n})}{n}}{1/n} = \lim_{n\to\infty} \cos(\frac{1}{n}) = 1$, a ~~positive~~ nonzero finite constant

So $\sum_{n=1}^{\infty} \frac{\cos(\frac{1}{n})}{n}$ Diverges.