71. There are only finitely many values of x where $Q(x) = 0$ (assuming that Q is not the zero polynomial). At all other values of x, $F(x)/Q(x) = G(x)/Q(x)$, so $F(x) = G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x. By continuity of F and G, the polynomials F and G must agree at those values of x too.

More explicitly: if a is a value of x such that $Q(a) = 0$, then $Q(x) \neq 0$ for all x sufficiently close to a. Thus,

$$\begin{aligned} F(a) &= \lim_{x\to a} F(x) && \text{[by continuity of } F\text{]} \\ &= \lim_{x\to a} G(x) && \text{[whenever } Q(x) \neq 0\text{]} \\ &= G(a) && \text{[by continuity of } G\text{]} \end{aligned}$$

72. Let $f(x) = ax^2 + bx + c$. We calculate the partial fraction decomposition of $\dfrac{f(x)}{x^2(x+1)^3}$. Since $f(0) = 1$, we must have $c = 1$, so $\dfrac{f(x)}{x^2(x+1)^3} = \dfrac{ax^2 + bx + 1}{x^2(x+1)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} + \dfrac{E}{(x+1)^3}$. Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have $A = C = 0$, so $ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$. Equating constant terms gives $B = 1$, then equating coefficients of x gives $3B = b \;\Rightarrow\; b = 3$. This is the quantity we are looking for, since $f'(0) = b$.

73. If $a \neq 0$ and n is a positive integer, then $f(x) = \dfrac{1}{x^n(x-a)} = \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \cdots + \dfrac{A_n}{x^n} + \dfrac{B}{x-a}$. Multiply both sides by $x^n(x-a)$ to get $1 = A_1x^{n-1}(x-a) + A_2x^{n-2}(x-a) + \cdots + A_n(x-a) + Bx^n$. Let $x = a$ in the last equation to get $1 = Ba^n \;\Rightarrow\; B = 1/a^n$. So

$$\begin{aligned} f(x) - \frac{B}{x-a} &= \frac{1}{x^n(x-a)} - \frac{1}{a^n(x-a)} = \frac{a^n - x^n}{x^na^n(x-a)} = -\frac{x^n - a^n}{a^nx^n(x-a)} \\ &= -\frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})}{a^nx^n(x-a)} \\ &= -\left(\frac{x^{n-1}}{a^nx^n} + \frac{x^{n-2}a}{a^nx^n} + \frac{x^{n-3}a^2}{a^nx^n} + \cdots + \frac{xa^{n-2}}{a^nx^n} + \frac{a^{n-1}}{a^nx^n}\right) \\ &= -\frac{1}{a^nx} - \frac{1}{a^{n-1}x^2} - \frac{1}{a^{n-2}x^3} - \cdots - \frac{1}{a^2x^{n-1}} - \frac{1}{ax^n} \end{aligned}$$

Thus, $f(x) = \dfrac{1}{x^n(x-a)} = -\dfrac{1}{a^nx} - \dfrac{1}{a^{n-1}x^2} - \cdots - \dfrac{1}{ax^n} + \dfrac{1}{a^n(x-a)}$.

7.5 Strategy for Integration

1. Let $u = \sin x$, so that $du = \cos x\,dx$. Then $\int \cos x(1 + \sin^2 x)\,dx = \int (1 + u^2)\,du = u + \frac{1}{3}u^3 + C = \sin x + \frac{1}{3}\sin^3 x + C$.

2. Let $u = 3x + 1$. Then $du = 3\,dx \;\Rightarrow$

$$\int_0^1 (3x+1)^{\sqrt{2}}\,dx = \int_1^4 u^{\sqrt{2}}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{1}{\sqrt{2}+1}\,u^{\sqrt{2}+1}\right]_1^4 = \frac{1}{3(\sqrt{2}+1)}\left(4^{\sqrt{2}+1} - 1\right)$$

3. $\displaystyle\int \frac{\sin x + \sec x}{\tan x}\,dx = \int \left(\frac{\sin x}{\tan x} + \frac{\sec x}{\tan x}\right)dx = \int (\cos x + \csc x)\,dx = \sin x + \ln|\csc x - \cot x| + C$

4. $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\sin^2 x \sin x}{\cos x}\,dx = \int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx = \int \frac{1-u^2}{u}(-du)$ $\quad\begin{bmatrix} u = \cos x \\ du = -\sin x\,dx \end{bmatrix}$

$= \int \left(u - \frac{1}{u}\right) du = \frac{1}{2}u^2 - \ln|u| + C = \frac{1}{2}\cos^2 x - \ln|\cos x| + C$

5. Let $u = t^2$. Then $du = 2t\,dt \Rightarrow$

$$\int \frac{t}{t^4+2}\,dt = \int \frac{1}{u^2+2}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C \text{ [by Formula 17]} = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right) + C$$

6. Let $u = 2x+1$. Then $du = 2\,dx \Rightarrow$

$$\int_0^1 \frac{x}{(2x+1)^3}\,dx = \int_1^3 \frac{(u-1)/2}{u^3}\left(\frac{1}{2}\,du\right) = \frac{1}{4}\int_1^3 \left(\frac{1}{u^2} - \frac{1}{u^3}\right) du = \frac{1}{4}\left[-\frac{1}{u} + \frac{1}{2u^2}\right]_1^3$$

$$= \tfrac{1}{4}\left[\left(-\tfrac{1}{3} + \tfrac{1}{18}\right) - \left(-1 + \tfrac{1}{2}\right)\right] = \tfrac{1}{4}\left(\tfrac{2}{9}\right) = \tfrac{1}{18}$$

7. Let $u = \arctan y$. Then $du = \dfrac{dy}{1+y^2} \Rightarrow \displaystyle\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = \left[e^u\right]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}$.

8. $\int t\sin t\,\cos t\,dt = \int t\cdot\frac{1}{2}(2\sin t\,\cos t)\,dt = \frac{1}{2}\int t\sin 2t\,dt$

$= \frac{1}{2}\left(-\frac{1}{2}t\cos 2t - \int -\frac{1}{2}\cos 2t\,dt\right)$ $\quad\begin{bmatrix} u = t, & dv = \sin 2t\,dt \\ du = dt, & v = -\frac{1}{2}\cos 2t \end{bmatrix}$

$= -\frac{1}{4}t\cos 2t + \frac{1}{4}\int \cos 2t\,dt = -\frac{1}{4}t\cos 2t + \frac{1}{8}\sin 2t + C$

9. $\int_1^3 r^4 \ln r\,dr$ $\quad\begin{bmatrix} u = \ln r, & dv = r^4\,dr, \\ du = \frac{dr}{r} & v = \frac{1}{5}r^5 \end{bmatrix}$ $= \left[\frac{1}{5}r^5\ln r\right]_1^3 - \int_1^3 \frac{1}{5}r^4\,dr = \frac{243}{5}\ln 3 - 0 - \left[\frac{1}{25}r^5\right]_1^3$

$= \frac{243}{5}\ln 3 - \left(\frac{243}{25} - \frac{1}{25}\right) = \frac{243}{5}\ln 3 - \frac{242}{25}$

10. $\dfrac{x-1}{x^2-4x-5} = \dfrac{x-1}{(x-5)(x+1)} = \dfrac{A}{x-5} + \dfrac{B}{x+1} \Rightarrow x - 1 = A(x+1) + B(x-5)$. Setting $x = -1$ gives $-2 = -6B$, so $B = \frac{1}{3}$. Setting $x = 5$ gives $4 = 6A$, so $A = \frac{2}{3}$. Now

$$\int_0^4 \frac{x-1}{x^2-4x-5}\,dx = \int_0^4 \left(\frac{2/3}{x-5} + \frac{1/3}{x+1}\right) dx = \left[\tfrac{2}{3}\ln|x-5| + \tfrac{1}{3}\ln|x+1|\right]_0^4$$

$$= \tfrac{2}{3}\ln 1 + \tfrac{1}{3}\ln 5 - \tfrac{2}{3}\ln 5 - \tfrac{1}{3}\ln 1 = -\tfrac{1}{3}\ln 5$$

11. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx = \int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right) du$ $\quad[u = x-2,\ du = dx]$

$= \frac{1}{2}\ln(u^2+1) + \tan^{-1} u + C = \frac{1}{2}\ln(x^2-4x+5) + \tan^{-1}(x-2) + C$

12. $\displaystyle\int \frac{x}{x^4+x^2+1}\,dx = \int \frac{\frac{1}{2}\,du}{u^2+u+1}$ $\quad\begin{bmatrix} u = x^2, \\ du = 2x\,dx \end{bmatrix}$ $\quad\displaystyle = \frac{1}{2}\int \frac{du}{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}$

$\displaystyle = \frac{1}{2}\int \frac{\frac{\sqrt{3}}{2}\,dv}{\frac{3}{4}(v^2+1)}$ $\quad\begin{bmatrix} u + \frac{1}{2} = \frac{\sqrt{3}}{2}v, \\ du = \frac{\sqrt{3}}{2}\,dv \end{bmatrix}$ $\quad\displaystyle = \frac{\sqrt{3}}{4}\cdot\frac{4}{3}\int \frac{dv}{v^2+1}$

$= \frac{1}{\sqrt{3}}\tan^{-1} v + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x^2 + \frac{1}{2}\right)\right) + C$

13. $\int \sin^5 t \cos^4 t\,dt = \int \sin^4 t \cos^4 t \sin t\,dt = \int (\sin^2 t)^2 \cos^4 t \sin t\,dt$

$= \int (1-\cos^2 t)^2 \cos^4 t \sin t\,dt = \int (1-u^2)^2 u^4 (-du)$ $\quad [u = \cos t,\ du = -\sin t\,dt]$

$= \int (-u^4 + 2u^6 - u^8)\,du = -\frac{1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C = -\frac{1}{5}\cos^5 t + \frac{2}{7}\cos^7 t - \frac{1}{9}\cos^9 t + C$

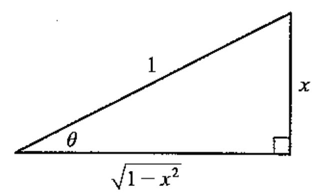
14. Let $u = 1 + x^2$, so that $du = 2x\,dx$. Then

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{x^2}{\sqrt{1+x^2}}\,(x\,dx) = \int \frac{u-1}{u^{1/2}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int (u^{1/2} - u^{-1/2})\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C$$

$$= \tfrac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C \quad \left[\text{or } \tfrac{1}{3}(x^2-2)\sqrt{1+x^2} + C\right]$$

15. Let $x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = \cos\theta\,d\theta$ and $(1-x^2)^{1/2} = \cos\theta$, so

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\,d\theta}{(\cos\theta)^3} = \int \sec^2\theta\,d\theta = \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C.$$



16. $$\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \qquad \begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{bmatrix}$$

$$= \int_0^{\pi/4} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{4} - \tfrac{1}{2}\right) - (0-0)\right] = \tfrac{\pi}{8} - \tfrac{1}{4}$$

17. $\int_0^\pi t\cos^2 t\,dt = \int_0^\pi t\left[\frac{1}{2}(1+\cos 2t)\right]dt = \frac{1}{2}\int_0^\pi t\,dt + \frac{1}{2}\int_0^\pi t\cos 2t\,dt$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}t^2\right]_0^\pi + \tfrac{1}{2}\left[\tfrac{1}{2}t\sin 2t\right]_0^\pi - \tfrac{1}{2}\int_0^\pi \tfrac{1}{2}\sin 2t\,dt \qquad \begin{bmatrix} u = t, & dv = \cos 2t\,dt \\ du = dt, & v = \tfrac{1}{2}\sin 2t \end{bmatrix}$$

$$= \tfrac{1}{4}\pi^2 + 0 - \tfrac{1}{4}\left[-\tfrac{1}{2}\cos 2t\right]_0^\pi = \tfrac{1}{4}\pi^2 + \tfrac{1}{8}(1-1) = \tfrac{1}{4}\pi^2$$

18. Let $u = \sqrt{t}$. Then $du = \dfrac{1}{2\sqrt{t}}\,dt \;\Rightarrow\; \displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt = \int_1^2 e^u\,(2\,du) = 2\left[e^u\right]_1^2 = 2(e^2 - e).$

19. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x} e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C.$

20. Since e^2 is a constant, $\int e^2\,dx = e^2 x + C.$

21. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Then $\int \arctan\sqrt{x}\,dx = \int \arctan t\,(2t\,dt) = I$. Now use parts with $u = \arctan t$, $dv = 2t\,dt \;\Rightarrow\; du = \dfrac{1}{1+t^2}\,dt$, $v = t^2$. Thus,

$$I = t^2\arctan t - \int \frac{t^2}{1+t^2}\,dt = t^2\arctan t - \int\left(1 - \frac{1}{1+t^2}\right)dt = t^2\arctan t - t + \arctan t + C$$

$$= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C \quad \left[\text{or } (x+1)\arctan\sqrt{x} - \sqrt{x} + C\right]$$

22. Let $u = 1 + (\ln x)^2$, so that $du = \dfrac{2\ln x}{x}\,dx$. Then

$$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\left(2\sqrt{u}\right) + C = \sqrt{1+(\ln x)^2} + C.$$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \Rightarrow$

$\int_0^1 \left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8 \cdot 2(u-1)\,du = 2\int_1^2 (u^9 - u^8)\,du = \left[\frac{1}{5}u^{10} - 2\cdot\frac{1}{9}u^9\right]_1^2 = \frac{1024}{5} - \frac{1024}{9} - \frac{1}{5} + \frac{2}{9} = \frac{4097}{45}$.

24. $$\int_0^4 \frac{6z+5}{2z+1}\,dz = \int_0^4 \frac{6z+3+2}{2z+1}\,dz = \int_0^4 \frac{3(2z+1)+2}{2z+1}\,dz = \int_0^4 \left(3 + \frac{2}{2z+1}\right) dz$$
$$= \Big[3z + \ln|2z+1|\Big]_0^4 = 12 + \ln 9$$

25. $\dfrac{3x^2-2}{x^2-2x-8} = 3 + \dfrac{6x+22}{(x-4)(x+2)} = 3 + \dfrac{A}{x-4} + \dfrac{B}{x+2} \Rightarrow 6x+22 = A(x+2) + B(x-4)$. Setting

$x = 4$ gives $46 = 6A$, so $A = \frac{23}{3}$. Setting $x = -2$ gives $10 = -6B$, so $B = -\frac{5}{3}$. Now

$$\int \frac{3x^2-2}{x^2-2x-8}\,dx = \int \left(3 + \frac{23/3}{x-4} - \frac{5/3}{x+2}\right) dx = 3x + \tfrac{23}{3}\ln|x-4| - \tfrac{5}{3}\ln|x+2| + C.$$

26. $$\int \frac{3x^2-2}{x^3-2x-8}\,dx = \int \frac{du}{u} \quad \begin{bmatrix} u = x^3-2x-8, \\ du = (3x^2-2)\,dx \end{bmatrix} \quad = \ln|u| + C = \ln\left|x^3-2x-8\right| + C$$

27. Let $u = 1 + e^x$, so that $du = e^x\,dx = (u-1)\,dx$. Then $\displaystyle\int \frac{1}{1+e^x}\,dx = \int \frac{1}{u}\cdot\frac{du}{u-1} = \int \frac{1}{u(u-1)}\,du = I$. Now

$\dfrac{1}{u(u-1)} = \dfrac{A}{u} + \dfrac{B}{u-1} \Rightarrow 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $I = \displaystyle\int \left(\frac{-1}{u} + \frac{1}{u-1}\right) du = -\ln|u| + \ln|u-1| + C = -\ln(1+e^x) + \ln e^x + C = x - \ln(1+e^x) + C$.

Another method: Multiply numerator and denominator by e^{-x} and let $u = e^{-x} + 1$. This gives the answer in the form $-\ln(e^{-x}+1) + C$.

28. $\int \sin\sqrt{at}\,dt = \int \sin u \cdot \frac{2}{a}u\,du \quad [u = \sqrt{at},\ u^2 = at,\ 2u\,du = a\,dt] \quad = \frac{2}{a}\int u \sin u\,du$

$= \frac{2}{a}[-u\cos u + \sin u] + C \quad$ [integration by parts] $\quad = -\frac{2}{a}\sqrt{at}\,\cos\sqrt{at} + \frac{2}{a}\sin\sqrt{at} + C$

$= -2\sqrt{\frac{t}{a}}\,\cos\sqrt{at} + \frac{2}{a}\sin\sqrt{at} + C$

29. Use integration by parts with $u = \ln\left(x + \sqrt{x^2-1}\right)$, $dv = dx \Rightarrow$

$$du = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) dx = \frac{1}{x+\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right) dx = \frac{1}{\sqrt{x^2-1}}\,dx,\ v = x. \text{ Then}$$

$$\int \ln\left(x+\sqrt{x^2-1}\right) dx = x\ln\left(x+\sqrt{x^2-1}\right) - \int \frac{x}{\sqrt{x^2-1}}\,dx = x\ln\left(x+\sqrt{x^2-1}\right) - \sqrt{x^2-1} + C.$$

30. $$|e^x - 1| = \begin{cases} e^x - 1 & \text{if } e^x - 1 \ge 0 \\ -(e^x-1) & \text{if } e^x - 1 < 0 \end{cases} = \begin{cases} e^x - 1 & \text{if } x \ge 0 \\ 1 - e^x & \text{if } x < 0 \end{cases}$$

Thus, $\int_{-1}^2 |e^x - 1|\,dx = \int_{-1}^0 (1-e^x)\,dx + \int_0^2 (e^x-1)\,dx = \Big[x - e^x\Big]_{-1}^0 + \Big[e^x - x\Big]_0^2$

$= (0-1) - (-1-e^{-1}) + (e^2-2) - (1-0) = e^2 + e^{-1} - 3$

31. As in Example 5,

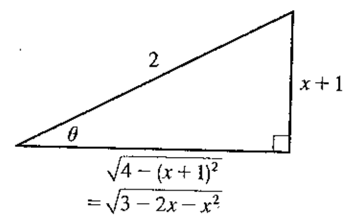
$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x - \sqrt{1-x^2} + C.$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

32. $$\int \frac{\sqrt{2x-1}}{2x+3}\,dx = \int \frac{u \cdot u\,du}{u^2+4} \quad \begin{bmatrix} u = \sqrt{2x-1},\, 2x+3 = u^2+4, \\ u^2 = 2x-1,\, u\,du = dx \end{bmatrix} = \int \left(1 - \frac{4}{u^2+4}\right) du$$

$$= u - 4 \cdot \tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right) + C = \sqrt{2x-1} - 2\tan^{-1}\left(\tfrac{1}{2}\sqrt{2x-1}\right) + C$$

33. $3 - 2x - x^2 = -(x^2+2x+1) + 4 = 4 - (x+1)^2$. Let $x + 1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$ and



$$\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \int \sqrt{4 - 4\sin^2\theta}\,2\cos\theta\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta = 2\int (1 + \cos 2\theta)\,d\theta$$

$$= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2 \cdot \frac{x+1}{2} \cdot \frac{\sqrt{3-2x-x^2}}{2} + C$$

$$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C$$

34. $$\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx = \int_{\pi/4}^{\pi/2} \left[\frac{(1+4\cos x/\sin x)}{(4-\cos x/\sin x)} \cdot \frac{\sin x}{\sin x}\right] dx = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4\cos x}{4\sin x - \cos x}\,dx$$

$$= \int_{3/\sqrt{2}}^{4} \frac{1}{u}\,du \quad \begin{bmatrix} u = 4\sin x - \cos x, \\ du = (4\cos x + \sin x)\,dx \end{bmatrix}$$

$$= \Big[\ln|u|\Big]_{3/\sqrt{2}}^{4} = \ln 4 - \ln\frac{3}{\sqrt{2}} = \ln\frac{4}{3/\sqrt{2}} = \ln\left(\frac{4}{3}\sqrt{2}\right)$$

35. Using product formula 2(c) in Section 7.2,

$\cos 2x\,\cos 6x = \frac{1}{2}[\cos(2x-6x) + \cos(2x+6x)] = \frac{1}{2}[\cos(-4x) + \cos 8x] = \frac{1}{2}(\cos 4x + \cos 8x)$. Thus,

$\int \cos 2x\,\cos 6x\,dx = \frac{1}{2}\int(\cos 4x + \cos 8x)\,dx = \frac{1}{2}\left(\frac{1}{4}\sin 4x + \frac{1}{8}\sin 8x\right) + C = \frac{1}{8}\sin 4x + \frac{1}{16}\sin 8x + C.$

36. The integrand is an odd function, so $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{x^2\tan x}{1+\cos^4 x}\,dx = 0$ [by 5.5.7(b)].

37. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta \;\Rightarrow\; \int_0^{\pi/4} \tan^3\theta\,\sec^2\theta\,d\theta = \int_0^1 u^3\,du = \left[\frac{1}{4}u^4\right]_0^1 = \frac{1}{4}$.

38. $$\int_{\pi/6}^{\pi/3} \frac{\sin\theta\,\cot\theta}{\sec\theta}\,d\theta = \int_{\pi/6}^{\pi/3} \cos^2\theta\,d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3} (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)\right] = \frac{1}{2}\left(\frac{\pi}{6}\right) = \frac{\pi}{12}$$

39. Let $u = \sec\theta$, so that $du = \sec\theta\,\tan\theta\,d\theta$. Then $\displaystyle\int \frac{\sec\theta\,\tan\theta}{\sec^2\theta - \sec\theta}\,d\theta = \int \frac{1}{u^2-u}\,du = \int \frac{1}{u(u-1)}\,du = I$. Now

$\dfrac{1}{u(u-1)} = \dfrac{A}{u} + \dfrac{B}{u-1} \;\Rightarrow\; 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $I = \displaystyle\int \left(\frac{-1}{u} + \frac{1}{u-1}\right) du = -\ln|u| + \ln|u-1| + C = \ln|\sec\theta - 1| - \ln|\sec\theta| + C$ [or $\ln|1-\cos\theta| + C$].

40. $4y^2 - 4y - 3 = (2y-1)^2 - 2^2$, so let $u = 2y - 1 \;\Rightarrow\; du = 2\,dy$. Thus,

$$\begin{aligned}\int \frac{dy}{\sqrt{4y^2-4y-3}} &= \int \frac{dy}{\sqrt{(2y-1)^2-2^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-2^2}} \\ &= \tfrac{1}{2}\ln\left|u + \sqrt{u^2-2^2}\right| \qquad \text{[by Formula 20 in the table in this section]} \\ &= \tfrac{1}{2}\ln\left|2y - 1 + \sqrt{4y^2-4y-3}\right| + C\end{aligned}$$

41. Let $u = \theta$, $dv = \tan^2\theta\,d\theta = (\sec^2\theta - 1)\,d\theta \;\Rightarrow\; du = d\theta$ and $v = \tan\theta - \theta$. So

$$\begin{aligned}\textstyle\int \theta\tan^2\theta\,d\theta &= \theta(\tan\theta - \theta) - \textstyle\int(\tan\theta - \theta)\,d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C \\ &= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C\end{aligned}$$

42. Let $u = \tan^{-1}x$, $dv = \dfrac{1}{x^2}\,dx \;\Rightarrow\; du = \dfrac{1}{1+x^2}\,dx$, $v = -\dfrac{1}{x}$. Then

$$I = \int \frac{\tan^{-1}x}{x^2}\,dx = -\frac{1}{x}\tan^{-1}x - \int\left(-\frac{1}{x(1+x^2)}\right)dx = -\frac{1}{x}\tan^{-1}x + \int\left(\frac{A}{x} + \frac{Bx+C}{1+x^2}\right)dx$$

$\dfrac{1}{x(1+x^2)} = \dfrac{A}{x} + \dfrac{Bx+C}{1+x^2} \;\Rightarrow\; 1 = A(1+x^2) + (Bx+C)x \;\Rightarrow\; 1 = (A+B)x^2 + Cx + A$, so $C = 0$, $A = 1$,

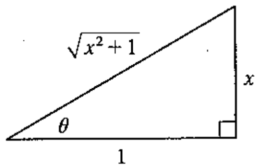
and $A + B = 0 \;\Rightarrow\; B = -1$. Thus,

$$\begin{aligned}I &= -\frac{1}{x}\tan^{-1}x + \int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx = -\frac{1}{x}\tan^{-1}x + \ln|x| - \frac{1}{2}\ln\left|1+x^2\right| + C \\ &= -\frac{\tan^{-1}x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C\end{aligned}$$

Or: Let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$. Then $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx = \int \frac{\theta}{\tan^2\theta}\sec^2\theta\,d\theta = \int \theta\csc^2\theta\,d\theta = I$. Now use parts

with $u = \theta$, $dv = \csc^2\theta\,d\theta \;\Rightarrow\; du = d\theta$, $v = -\cot\theta$. Thus,

$$\begin{aligned}I &= -\theta\cot\theta - \textstyle\int(-\cot\theta)\,d\theta = -\theta\cot\theta + \ln|\sin\theta| + C \\ &= -\tan^{-1}x\cdot\frac{1}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C = -\frac{\tan^{-1}x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C\end{aligned}$$



43. Let $u = \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{u}{1+u^6}\,(2u\,du) = 2\int \frac{u^2}{1+(u^3)^2}\,du = 2\int \frac{1}{1+t^2}\left(\frac{1}{3}\,dt\right) \qquad \begin{bmatrix} t = u^3 \\ dt = 3u^2\,du \end{bmatrix}$$
$$= \tfrac{2}{3}\tan^{-1} t + C = \tfrac{2}{3}\tan^{-1} u^3 + C = \tfrac{2}{3}\tan^{-1}(x^{3/2}) + C$$

Another method: Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \frac{3}{2}x^{1/2}\,dx \;\Rightarrow\; \sqrt{x}\,dx = \frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1} u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

44. Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$, $2u\,du = e^x\,dx = (u^2-1)\,dx$, and $dx = \dfrac{2u}{u^2-1}du$, so

$$\int \sqrt{1+e^x}\,dx = \int u \cdot \frac{2u}{u^2-1}\,du = \int \frac{2u^2}{u^2-1}\,du = \int\left(2 + \frac{2}{u^2-1}\right)du = \int\left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du$$
$$= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln\left(\sqrt{1+e^x}-1\right) - \ln\left(\sqrt{1+e^x}+1\right) + C$$

45. Let $t = x^3$. Then $dt = 3x^2\,dx \;\Rightarrow\; I = \int x^5 e^{-x^3}\,dx = \frac{1}{3}\int te^{-t}\,dt$. Now integrate by parts with $u = t$, $dv = e^{-t}\,dt$:

$I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\,dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}(x^3+1) + C.$

46. Use integration by parts with $u = (x-1)e^x$, $dv = \dfrac{1}{x^2}\,dx \;\Rightarrow\; du = [(x-1)e^x + e^x]\,dx = xe^x\,dx$, $v = -\dfrac{1}{x}$. Then

$$\int \frac{(x-1)e^x}{x^2}\,dx = (x-1)e^x\left(-\frac{1}{x}\right) - \int -e^x\,dx = -e^x + \frac{e^x}{x} + e^x + C = \frac{e^x}{x} + C.$$

47. Let $u = x - 1$, so that $du = dx$. Then

$$\int x^3(x-1)^{-4}\,dx = \int (u+1)^3 u^{-4}\,du = \int (u^3 + 3u^2 + 3u + 1)u^{-4}\,du = \int (u^{-1} + 3u^{-2} + 3u^{-3} + u^{-4})\,du$$
$$= \ln|u| - 3u^{-1} - \tfrac{3}{2}u^{-2} - \tfrac{1}{3}u^{-3} + C = \ln|x-1| - 3(x-1)^{-1} - \tfrac{3}{2}(x-1)^{-2} - \tfrac{1}{3}(x-1)^{-3} + C$$

48. Let $u = \sqrt{1-x^2}$, so $u^2 = 1 - x^2$, and $2u\,du = -2x\,dx$. Then $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx = \int_1^0 \sqrt{2-u}\,(-u\,du)$.

Now let $v = \sqrt{2-u}$, so $v^2 = 2 - u$, and $2v\,dv = -du$. Thus,

$$\int_1^0 \sqrt{2-u}\,(-u\,du) = \int_1^{\sqrt{2}} v(2-v^2)\,(2v\,dv) = \int_1^{\sqrt{2}} (4v^2 - 2v^4)\,dv = \left[\tfrac{4}{3}v^3 - \tfrac{2}{5}v^5\right]_1^{\sqrt{2}}$$
$$= \left(\tfrac{8}{3}\sqrt{2} - \tfrac{8}{5}\sqrt{2}\right) - \left(\tfrac{4}{3} - \tfrac{2}{5}\right) = \tfrac{16}{15}\sqrt{2} - \tfrac{14}{15}$$

49. Let $u = \sqrt{4x+1} \;\Rightarrow\; u^2 = 4x + 1 \;\Rightarrow\; 2u\,du = 4\,dx \;\Rightarrow\; dx = \frac{1}{2}u\,du$. So

$$\int \frac{1}{x\sqrt{4x+1}}\,dx = \int \frac{\frac{1}{2}u\,du}{\frac{1}{4}(u^2-1)\,u} = 2\int \frac{du}{u^2-1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right| + C \qquad \text{[by Formula 19]}$$
$$= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$$

50. As in Exercise 49, let $u = \sqrt{4x+1}$. Then $\displaystyle\int \frac{dx}{x^2\sqrt{4x+1}} = \int \frac{\frac{1}{2}u\,du}{\left[\frac{1}{4}(u^2-1)\right]^2 u} = 8\int \frac{du}{(u^2-1)^2}$. Now

$$\frac{1}{(u^2-1)^2} = \frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2} \quad\Rightarrow$$

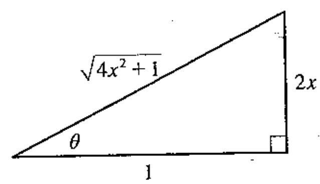
$1 = A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2$. $u = 1 \Rightarrow D = \frac{1}{4}$, $u = -1 \Rightarrow B = \frac{1}{4}$.

Equating coefficients of u^3 gives $A + C = 0$, and equating coefficients of 1 gives $1 = A + B - C + D \Rightarrow$ $1 = A + \frac{1}{4} - C + \frac{1}{4} \Rightarrow \frac{1}{2} = A - C$. So $A = \frac{1}{4}$ and $C = -\frac{1}{4}$. Therefore,

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{4x+1}} &= 8\int\left[\frac{1/4}{u+1} + \frac{1/4}{(u+1)^2} + \frac{-1/4}{u-1} + \frac{1/4}{(u-1)^2}\right]du \\ &= \int\left[\frac{2}{u+1} + 2(u+1)^{-2} - \frac{2}{u-1} + 2(u-1)^{-2}\right]du \\ &= 2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1} + C \\ &= 2\ln\left(\sqrt{4x+1}+1\right) - \frac{2}{\sqrt{4x+1}+1} - 2\ln\left|\sqrt{4x+1}-1\right| - \frac{2}{\sqrt{4x+1}-1} + C\end{aligned}$$

51. Let $2x = \tan\theta \Rightarrow x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\, d\theta$, $\sqrt{4x^2+1} = \sec\theta$, so

$$\begin{aligned}\int \frac{dx}{x\sqrt{4x^2+1}} &= \int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\frac{1}{2}\tan\theta\,\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\, d\theta \\ &= -\ln|\csc\theta + \cot\theta| + C \qquad [\text{or } \ln|\csc\theta - \cot\theta| + C] \\ &= -\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C \qquad \left[\text{or } \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C\right]\end{aligned}$$



52. Let $u = x^2$. Then $du = 2x\,dx \Rightarrow$

$$\begin{aligned}\int \frac{dx}{x(x^4+1)} &= \int \frac{x\,dx}{x^2(x^4+1)} = \frac{1}{2}\int \frac{du}{u(u^2+1)} = \frac{1}{2}\int\left[\frac{1}{u} - \frac{u}{u^2+1}\right]du = \frac{1}{2}\ln|u| - \frac{1}{4}\ln(u^2+1) + C \\ &= \frac{1}{2}\ln(x^2) - \frac{1}{4}\ln(x^4+1) + C = \frac{1}{4}\left[\ln(x^4) - \ln(x^4+1)\right] + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C\end{aligned}$$

Or: Write $I = \int \frac{x^3\,dx}{x^4(x^4+1)}$ and let $u = x^4$.

53. $\int x^2\sinh(mx)dx = \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\int x\cosh(mx)\,dx$ $\qquad\left[\begin{matrix} u = x^2, & dv = \sinh(mx)\,dx, \\ du = 2x\,dx & v = \frac{1}{m}\cosh(mx)\end{matrix}\right]$

$$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\left(\frac{1}{m}x\sinh(mx) - \frac{1}{m}\int\sinh(mx)\,dx\right) \qquad \left[\begin{matrix} U = x, & dV = \cosh(mx)\,dx, \\ dU = dx & V = \frac{1}{m}\sinh(mx)\end{matrix}\right]$$

$$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m^2}x\sinh(mx) + \frac{2}{m^3}\cosh(mx) + C$$

54. $\int(x+\sin x)^2\,dx = \int\left(x^2 + 2x\sin x + \sin^2 x\right)dx = \frac{1}{3}x^3 + 2(\sin x - x\cos x) + \frac{1}{2}(x - \sin x\cos x) + C$
$= \frac{1}{3}x^3 + \frac{1}{2}x + 2\sin x - \frac{1}{2}\sin x\cos x - 2x\cos x + C$

55. Let $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Then $\int \frac{dx}{x + x\sqrt{x}} = \int \frac{2u\,du}{u^2 + u^2\cdot u} = \int \frac{2}{u(1+u)}\,du = I$.

Now $\frac{2}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u} \Rightarrow 2 = A(1+u) + Bu$. Set $u = -1$ to get $2 = -B$, so $B = -2$. Set $u = 0$ to get $2 = A$.

Thus, $I = \int\left(\frac{2}{u} - \frac{2}{1+u}\right)du = 2\ln|u| - 2\ln|1+u| + C = 2\ln\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$.

56. Let $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Then

$$\int \frac{dx}{\sqrt{x} + x\sqrt{x}} = \int \frac{2u\,du}{u + u^2 \cdot u} = \int \frac{2}{1+u^2}\,du = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C.$$

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \;\Rightarrow$

$$\int x\,\sqrt[3]{x+c}\,dx = \int (u^3 - c)u \cdot 3u^2\,du = 3\int (u^6 - cu^3)\,du = \tfrac{3}{7}u^7 - \tfrac{3}{4}cu^4 + C = \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

58. Let $t = \sqrt{x^2-1}$. Then $dt = \left(x/\sqrt{x^2-1}\right)dx$, $x^2 - 1 = t^2$, $x = \sqrt{t^2+1}$, so

$$I = \int \frac{x\,\ln x}{\sqrt{x^2-1}}\,dx = \int \ln\sqrt{t^2+1}\,dt = \tfrac{1}{2}\int \ln(t^2+1)\,dt.$$ Now use parts with $u = \ln(t^2+1)$, $dv = dt$:

$$\begin{aligned} I &= \tfrac{1}{2}t\ln(t^2+1) - \int \frac{t^2}{t^2+1}\,dt = \tfrac{1}{2}t\ln(t^2+1) - \int\left[1 - \frac{1}{t^2+1}\right]dt \\ &= \tfrac{1}{2}t\ln(t^2+1) - t + \tan^{-1}t + C = \sqrt{x^2-1}\,\ln x - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1} + C \end{aligned}$$

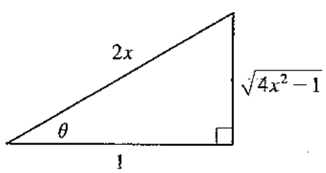
Another method: First integrate by parts with $u = \ln x$, $dv = \left(x/\sqrt{x^2-1}\right)dx$ and then use substitution $\left(x = \sec\theta \text{ or } u = \sqrt{x^2-1}\right)$.

59. Let $u = \sin x$, so that $du = \cos x\,dx$. Then

$$\begin{aligned} \int \cos x\,\cos^3(\sin x)\,dx &= \int \cos^3 u\,du = \int \cos^2 u\,\cos u\,du = \int (1 - \sin^2 u)\cos u\,du \\ &= \int (\cos u - \sin^2 u\,\cos u)\,du = \sin u - \tfrac{1}{3}\sin^3 u + C = \sin(\sin x) - \tfrac{1}{3}\sin^3(\sin x) + C \end{aligned}$$

60. Let $2x = \sec\theta$, so that $2\,dx = \sec\theta\,\tan\theta\,d\theta$. Then

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{4x^2-1}} &= \int \frac{\frac{1}{2}\sec\theta\,\tan\theta\,d\theta}{\frac{1}{4}\sec^2\theta\,\sqrt{\sec^2\theta - 1}} = \int \frac{2\tan\theta\,d\theta}{\sec\theta\,\tan\theta} \\ &= 2\int \cos\theta\,d\theta = 2\sin\theta + C \\ &= 2\cdot\frac{\sqrt{4x^2-1}}{2x} + C = \frac{\sqrt{4x^2-1}}{x} + C \end{aligned}$$



61. $$\begin{aligned} \int \frac{d\theta}{1+\cos\theta} &= \int \left(\frac{1}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}\right)d\theta = \int \frac{1-\cos\theta}{1-\cos^2\theta}\,d\theta = \int \frac{1-\cos\theta}{\sin^2\theta}\,d\theta = \int\left(\frac{1}{\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta}\right)d\theta \\ &= \int (\csc^2\theta - \cot\theta\,\csc\theta)\,d\theta = -\cot\theta + \csc\theta + C \end{aligned}$$

Another method: Use the substitutions in Exercise 7.4.59.

$$\int \frac{d\theta}{1+\cos\theta} = \int \frac{2/(1+t^2)\,dt}{1 + (1-t^2)/(1+t^2)} = \int \frac{2\,dt}{(1+t^2) + (1-t^2)} = \int dt = t + C = \tan\left(\frac{\theta}{2}\right) + C$$

62. $$\begin{aligned} \int \frac{d\theta}{1+\cos^2\theta} &= \int \frac{(1/\cos^2\theta)\,d\theta}{(1+\cos^2\theta)/\cos^2\theta} = \int \frac{\sec^2\theta}{\sec^2\theta + 1}\,d\theta = \int \frac{\sec^2\theta}{\tan^2\theta + 2}\,d\theta = \int \frac{1}{u^2+2}\,du \quad \left[\begin{array}{l} u = \tan\theta, \\ du = \sec^2\theta\,d\theta \end{array}\right] \\ &= \int \frac{1}{u^2 + \left(\sqrt{2}\right)^2}\,du = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan\theta}{\sqrt{2}}\right) + C \end{aligned}$$

63. Let $y = \sqrt{x}$ so that $dy = \dfrac{1}{2\sqrt{x}}\,dx \;\Rightarrow\; dx = 2\sqrt{x}\,dy = 2y\,dy$. Then

$$\begin{aligned}\int \sqrt{x}\,e^{\sqrt{x}}\,dx &= \int ye^{y}(2y\,dy) = \int 2y^2e^y\,dy \qquad \begin{bmatrix} u = 2y^2, & dv = e^y\,dy, \\ du = 4y\,dy & v = e^y \end{bmatrix} \\ &= 2y^2e^y - \int 4ye^y\,dy \qquad \begin{bmatrix} U = 4y, & dV = e^y\,dy, \\ dU = 4\,dy & V = e^y \end{bmatrix} \\ &= 2y^2e^y - \left(4ye^y - \textstyle\int 4e^y\,dy\right) = 2y^2e^y - 4ye^y + 4e^y + C \\ &= 2(y^2 - 2y + 2)e^y + C = 2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C\end{aligned}$$

64. Let $u = \sqrt{x} + 1$, so that $x = (u-1)^2$ and $dx = 2(u-1)\,du$. Then

$$\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx = \int \frac{2(u-1)\,du}{\sqrt{u}} = \int (2u^{1/2} - 2u^{-1/2})\,du = \frac{4}{3}u^{3/2} - 4u^{1/2} + C = \frac{4}{3}(\sqrt{x}+1)^{3/2} - 4\sqrt{\sqrt{x}+1} + C.$$

65. Let $u = \cos^2 x$, so that $du = 2\cos x\,(-\sin x)\,dx$. Then

$$\int \frac{\sin 2x}{1 + \cos^4 x}\,dx = \int \frac{2\sin x\,\cos x}{1 + (\cos^2 x)^2}\,dx = \int \frac{1}{1+u^2}\,(-du) = -\tan^{-1} u + C = -\tan^{-1}(\cos^2 x) + C.$$

66. Let $u = \tan x$. Then

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)\,dx}{\sin x\,\cos x} = \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x}\,\sec^2 x\,dx = \int_1^{\sqrt{3}} \frac{\ln u}{u}\,du = \left[\tfrac{1}{2}(\ln u)^2\right]_1^{\sqrt{3}} = \tfrac{1}{2}\left(\ln\sqrt{3}\right)^2 = \tfrac{1}{8}(\ln 3)^2.$$

67. $$\begin{aligned}\int \frac{dx}{\sqrt{x+1} + \sqrt{x}} &= \int \left(\frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}}\right) dx = \int \left(\sqrt{x+1} - \sqrt{x}\right) dx \\ &= \tfrac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C\end{aligned}$$

68. $$\int \frac{x^2}{x^6 + 3x^3 + 2}\,dx = \int \frac{x^2\,dx}{(x^3+1)(x^3+2)} = \int \frac{\frac{1}{3}\,du}{(u+1)(u+2)} \qquad \begin{bmatrix} u = x^3, \\ du = 3x^2\,dx \end{bmatrix}.$$

Now $\dfrac{1}{(u+1)(u+2)} = \dfrac{A}{u+1} + \dfrac{B}{u+2} \;\Rightarrow\; 1 = A(u+2) + B(u+1)$. Setting $u = -2$ gives $B = -1$. Setting $u = -1$ gives $A = 1$. Thus,

$$\begin{aligned}\frac{1}{3}\int \frac{du}{(u+1)(u+2)} &= \frac{1}{3}\int \left(\frac{1}{u+1} - \frac{1}{u+2}\right) du = \frac{1}{3}\ln|u+1| - \frac{1}{3}\ln|u+2| + C \\ &= \tfrac{1}{3}\ln|x^3+1| - \tfrac{1}{3}\ln|x^3+2| + C\end{aligned}$$

69. Let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$, $x = \sqrt{3} \;\Rightarrow\; \theta = \frac{\pi}{3}$, and $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Then

$$\begin{aligned}\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx &= \int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan^2\theta}\,\sec^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sec\theta\,(\tan^2\theta + 1)}{\tan^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \left(\frac{\sec\theta\,\tan^2\theta}{\tan^2\theta} + \frac{\sec\theta}{\tan^2\theta}\right) d\theta \\ &= \int_{\pi/4}^{\pi/3} (\sec\theta + \csc\theta\,\cot\theta)\,d\theta = \Big[\ln|\sec\theta + \tan\theta| - \csc\theta\Big]_{\pi/4}^{\pi/3} \\ &= \left(\ln\left|2+\sqrt{3}\right| - \tfrac{2}{\sqrt{3}}\right) - \left(\ln\left|\sqrt{2}+1\right| - \sqrt{2}\right) = \sqrt{2} - \tfrac{2}{\sqrt{3}} + \ln\left(2+\sqrt{3}\right) - \ln\left(1+\sqrt{2}\right)\end{aligned}$$

70. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$ $\Rightarrow$

$$\int \frac{dx}{1 + 2e^x - e^{-x}} = \int \frac{du/u}{1 + 2u - 1/u} = \int \frac{du}{2u^2 + u - 1} = \int \left[\frac{2/3}{2u - 1} - \frac{1/3}{u + 1}\right] du$$
$$= \tfrac{1}{3} \ln|2u - 1| - \tfrac{1}{3} \ln|u + 1| + C = \tfrac{1}{3} \ln|(2e^x - 1)/(e^x + 1)| + C$$

71. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$ $\Rightarrow$

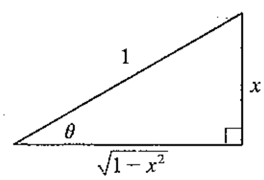
$$\int \frac{e^{2x}}{1 + e^x}\, dx = \int \frac{u^2}{1 + u} \frac{du}{u} = \int \frac{u}{1 + u}\, du = \int \left(1 - \frac{1}{1 + u}\right) du = u - \ln|1 + u| + C = e^x - \ln(1 + e^x) + C.$$

72. Use parts with $u = \ln(x + 1)$, $dv = dx/x^2$:

$$\int \frac{\ln(x + 1)}{x^2}\, dx = -\frac{1}{x} \ln(x + 1) + \int \frac{dx}{x(x + 1)} = -\frac{1}{x} \ln(x + 1) + \int \left[\frac{1}{x} - \frac{1}{x + 1}\right] dx$$
$$= -\frac{1}{x} \ln(x + 1) + \ln|x| - \ln(x + 1) + C = -\left(1 + \frac{1}{x}\right) \ln(x + 1) + \ln|x| + C$$

73. Let $\theta = \arcsin x$, so that $d\theta = \dfrac{1}{\sqrt{1 - x^2}}\, dx$ and $x = \sin\theta$. Then

$$\int \frac{x + \arcsin x}{\sqrt{1 - x^2}}\, dx = \int (\sin\theta + \theta)\, d\theta = -\cos\theta + \tfrac{1}{2}\theta^2 + C$$
$$= -\sqrt{1 - x^2} + \tfrac{1}{2}(\arcsin x)^2 + C$$



74. $$\int \frac{4^x + 10^x}{2^x}\, dx = \int \left(\frac{4^x}{2^x} + \frac{10^x}{2^x}\right) dx = \int (2^x + 5^x)\, dx = \frac{2^x}{\ln 2} + \frac{5^x}{\ln 5} + C$$

75. $$\frac{1}{(x - 2)(x^2 + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 4} \quad\Rightarrow\quad 1 = A(x^2 + 4) + (Bx + C)(x - 2) = (A + B)x^2 + (C - 2B)x + (4A - 2C).$$

So $0 = A + B = C - 2B$, $1 = 4A - 2C$. Setting $x = 2$ gives $A = \frac{1}{8}$ $\Rightarrow$ $B = -\frac{1}{8}$ and $C = -\frac{1}{4}$. So

$$\int \frac{1}{(x - 2)(x^2 + 4)}\, dx = \int \left(\frac{\frac{1}{8}}{x - 2} + \frac{-\frac{1}{8}x - \frac{1}{4}}{x^2 + 4}\right) dx = \frac{1}{8}\int \frac{dx}{x - 2} - \frac{1}{16}\int \frac{2x\, dx}{x^2 + 4} - \frac{1}{4}\int \frac{dx}{x^2 + 4}$$
$$= \tfrac{1}{8} \ln|x - 2| - \tfrac{1}{16} \ln(x^2 + 4) - \tfrac{1}{8} \tan^{-1}(x/2) + C$$

76. Let $u = 2 + \sqrt{x}$, so that $du = \dfrac{1}{2\sqrt{x}}\, dx$. Then

$$\int \frac{dx}{\sqrt{x}\,(2 + \sqrt{x})^4} = \int \frac{2\, du}{u^4} = 2\int u^{-4}\, du = -\frac{2}{3}u^{-3} + C = -\frac{2}{3\,(2 + \sqrt{x})^3} + C.$$

77. Let $y = \sqrt{1+e^x}$, so that $y^2 = 1 + e^x$, $2y\,dy = e^x\,dx$, $e^x = y^2 - 1$, and $x = \ln(y^2 - 1)$. Then

$$\begin{aligned}
\int \frac{xe^x}{\sqrt{1+e^x}}\,dx &= \int \frac{\ln(y^2-1)}{y}(2y\,dy) = 2\int [\ln(y+1) + \ln(y-1)]\,dy \\
&= 2[(y+1)\ln(y+1) - (y+1) + (y-1)\ln(y-1) - (y-1)] + C \qquad \text{[by Example 7.1.2]} \\
&= 2[y\ln(y+1) + \ln(y+1) - y - 1 + y\ln(y-1) - \ln(y-1) - y + 1] + C \\
&= 2[y(\ln(y+1) + \ln(y-1)) + \ln(y+1) - \ln(y-1) - 2y] + C \\
&= 2\left[y\ln(y^2-1) + \ln\frac{y+1}{y-1} - 2y\right] + C = 2\left[\sqrt{1+e^x}\,\ln(e^x) + \ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 2\sqrt{1+e^x}\right] + C \\
&= 2x\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 4\sqrt{1+e^x} + C = 2(x-2)\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} + C
\end{aligned}$$

78. $\dfrac{1+\sin x}{1-\sin x} = \dfrac{1+\sin x}{1-\sin x}\cdot\dfrac{1+\sin x}{1+\sin x} = \dfrac{1+2\sin x+\sin^2 x}{1-\sin^2 x} = \dfrac{1+2\sin x+\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} + \dfrac{2\sin x}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x}$

$= \sec^2 x + 2\sec x\,\tan x + \tan^2 x = \sec^2 x + 2\sec x\,\tan x + \sec^2 x - 1 = 2\sec^2 x + 2\sec x\,\tan x - 1$

Thus, $\displaystyle\int \frac{1+\sin x}{1-\sin x}\,dx = \int (2\sec^2 x + 2\sec x\,\tan x - 1)\,dx = 2\tan x + 2\sec x - x + C$

79. Let $u = x$, $dv = \sin^2 x\cos x\,dx \;\Rightarrow\; du = dx$, $v = \frac{1}{3}\sin^3 x$. Then

$$\begin{aligned}
\int x\sin^2 x\,\cos x\,dx &= \tfrac{1}{3}x\sin^3 x - \int \tfrac{1}{3}\sin^3 x\,dx = \tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\int(1-\cos^2 x)\sin x\,dx \\
&= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int(1-y^2)\,dy \qquad \begin{bmatrix} u = \cos x, \\ du = -\sin x\,dx \end{bmatrix} \\
&= \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}y - \tfrac{1}{9}y^3 + C = \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}\cos x - \tfrac{1}{9}\cos^3 x + C
\end{aligned}$$

80. $$\begin{aligned}
\int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\,dx &= \int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\cdot\frac{2\cos x}{2\cos x}\,dx = \int \frac{2\cos 2x}{2\sin x\cos x + 2}\,dx \\
&= \int \frac{2\cos 2x}{\sin 2x + 2}\,dx = \int \frac{1}{u}\,du \qquad \begin{bmatrix} u = \sin 2x + 2, \\ du = 2\cos 2x\,dx \end{bmatrix} \\
&= \ln|u| + C = \ln|\sin 2x + 2| + C = \ln(\sin 2x + 2) + C
\end{aligned}$$

81. $$\begin{aligned}
\int \sqrt{1-\sin x}\,dx &= \int \sqrt{\frac{1-\sin x}{1}\cdot\frac{1+\sin x}{1+\sin x}}\,dx = \int \sqrt{\frac{1-\sin^2 x}{1+\sin x}}\,dx \\
&= \int \sqrt{\frac{\cos^2 x}{1+\sin x}}\,dx = \int \frac{\cos x\,dx}{\sqrt{1+\sin x}} \qquad [\text{assume } \cos x > 0] \\
&= \int \frac{du}{\sqrt{u}} \qquad \begin{bmatrix} u = 1 + \sin x, \\ du = \cos x\,dx \end{bmatrix} \\
&= 2\sqrt{u} + C = 2\sqrt{1+\sin x} + C
\end{aligned}$$

Another method: Let $u = \sin x$ so that $du = \cos x\,dx = \sqrt{1-\sin^2 x}\,dx = \sqrt{1-u^2}\,dx$. Then

$$\int \sqrt{1-\sin x}\,dx = \int \sqrt{1-u}\left(\frac{du}{\sqrt{1-u^2}}\right) = \int \frac{1}{\sqrt{1+u}}\,du = 2\sqrt{1+u} + C = 2\sqrt{1+\sin x} + C.$$

82. $\displaystyle\int \frac{\sin x\,\cos x}{\sin^4 x+\cos^4 x}\,dx = \int \frac{\sin x\,\cos x}{(\sin^2 x)^2+(\cos^2 x)^2}\,dx = \int \frac{\sin x\,\cos x}{(\sin^2 x)^2+(1-\sin^2 x)^2}\,dx$

$$= \int \frac{1}{u^2+(1-u)^2}\left(\frac{1}{2}\,du\right) \quad \begin{bmatrix} u=\sin^2 x, \\ du = 2\sin x\,\cos x\,dx \end{bmatrix}$$

$$= \int \frac{1}{4u^2-4u+2}\,du = \int \frac{1}{(4u^2-4u+1)+1}\,du$$

$$= \int \frac{1}{(2u-1)^2+1}\,du = \frac{1}{2}\int \frac{1}{y^2+1}\,dy \quad \begin{bmatrix} y=2u-1, \\ dy = 2\,du \end{bmatrix}$$

$$= \tfrac{1}{2}\tan^{-1} y + C = \tfrac{1}{2}\tan^{-1}(2u-1)+C = \tfrac{1}{2}\tan^{-1}(2\sin^2 x-1)+C$$

Another solution: $\displaystyle\int \frac{\sin x\,\cos x}{\sin^4 x+\cos^4 x}\,dx = \int \frac{(\sin x\,\cos x)/\cos^4 x}{(\sin^4 x+\cos^4 x)/\cos^4 x}\,dx = \int \frac{\tan x\,\sec^2 x}{\tan^4 x+1}\,dx$

$$= \int \frac{1}{u^2+1}\left(\frac{1}{2}\,du\right) \quad \begin{bmatrix} u=\tan^2 x, \\ du = 2\tan x\,\sec^2 x\,dx \end{bmatrix}$$

$$= \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1}(\tan^2 x)+C$$

83. The function $y = 2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int (2x^2+1)e^{x^2}\,dx = \int 2x^2e^{x^2}\,dx + \int e^{x^2}\,dx = \int x\left(2xe^{x^2}\right)dx + \int e^{x^2}\,dx$$

$$= xe^{x^2} - \int e^{x^2}\,dx + \int e^{x^2}\,dx \quad \begin{bmatrix} u=x, & dv = 2xe^{x^2}\,dx, \\ du = dx & v = e^{x^2} \end{bmatrix} \quad = xe^{x^2}+C$$

84. (a) $\displaystyle\int_1^2 \frac{e^x}{x}\,dx = \int_0^{\ln 2} \frac{e^{e^t}}{e^t}\,e^t\,dt \quad \begin{bmatrix} x=e^t, \\ dx = e^t dt \end{bmatrix} \quad = \int_0^{\ln 2} e^{e^t}\,dt = F(\ln 2)$

(b) $\displaystyle\int_2^3 \frac{1}{\ln x}\,dx = \int_{\ln 2}^{\ln 3} \frac{1}{u}\,(e^u\,du) \quad \begin{bmatrix} u=\ln x, \\ du = \frac{1}{x}\,dx \end{bmatrix} \quad = \int_{\ln\ln 2}^{\ln\ln 3} \frac{e^{e^v}}{e^v}\,e^v\,dv \quad \begin{bmatrix} u=e^v, \\ du = e^v\,dv \end{bmatrix}$

$$= \int_{\ln\ln 2}^{0} e^{e^v}\,dv + \int_0^{\ln\ln 3} e^{e^v}\,dv \quad [\text{note that } \ln\ln 2 < 0]$$

$$= \int_0^{\ln\ln 3} e^{e^v}\,dv - \int_0^{\ln\ln 2} e^{e^v}\,dv = F(\ln\ln 3) - F(\ln\ln 2)$$

Another method: Substitute $x = e^{e^t}$ in the original integral.

7.6 Integration Using Tables and Computer Algebra Systems

Keep in mind that there are several ways to approach many of these exercises, and different methods can lead to different forms of the answer.

1. $\displaystyle\int_0^{\pi/2} \cos 5x\,\cos 2x\,dx \overset{80}{=} \left[\frac{\sin(5-2)x}{2(5-2)} + \frac{\sin(5+2)x}{2(5+2)}\right]_0^{\pi/2} \quad \begin{bmatrix} a=5, \\ b=2 \end{bmatrix}$

$$= \left[\frac{\sin 3x}{6} + \frac{\sin 7x}{14}\right]_0^{\pi/2} = \left(-\frac{1}{6}-\frac{1}{14}\right) - 0 = \frac{-7-3}{42} = -\frac{5}{21}$$

2. $\displaystyle\int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \sqrt{2\left(\tfrac{1}{2}\right)x - x^2}\,dx \overset{113}{=} \left[\frac{x-\frac{1}{2}}{2}\sqrt{2\left(\tfrac{1}{2}\right)x-x^2} + \frac{\left(\frac{1}{2}\right)^2}{2}\cos^{-1}\left(\frac{\frac{1}{2}-x}{\frac{1}{2}}\right)\right]_0^1$

$$= \left[\frac{2x-1}{4}\sqrt{x-x^2} + \frac{1}{8}\cos^{-1}(1-2x)\right]_0^1 = \left(0+\frac{1}{8}\cdot\pi\right) - \left(0+\frac{1}{8}\cdot 0\right) = \frac{1}{8}\pi$$