KEY

Math 126: Midterm Exam

You have 90 minutes to complete this closed-book, closed-notes, closed-colleague exam. You may use a calculator for arithmetic only (i.e., trigonometry, exponential, and logarithmic functions, but no Calculus and no Graphing). Please sign the following statement when you've finished.

> I certify that the work on this exam is entirely my own. I have not used a calculator for purposes other than arithmetic. I have not discussed the specific contents of this exam with anyone else prior to my taking it.
>
> *Signature:*

1. Tell whether each series converges conditionally, converges absolutely, or diverges. Be sure to use an appropriate test to justify your answers.

   (a)

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

$n > \ln(n)$ for large enough n

Thus $\frac{1}{n} < \frac{1}{\ln(n)}$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ Diverges, $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ Diverges

(b)

$$\sum_{n=0}^{\infty} \frac{(-1)^n(n^3+1)}{\sqrt{n^7+2}} \sim \sum (-1)^n \frac{n^3}{n^{7/2}} = \sum \frac{(-1)^n}{\sqrt{n}}$$

note: $\lim_{n\to\infty} \frac{n^3+1}{\sqrt{n^7+2}} \cdot \frac{n^{7/2}}{n^3} = \lim_{n\to\infty} \frac{n^3+1}{n} \cdot \frac{n^{7/2}}{\sqrt{n^7+2}} = 1 \cdot 1 = 1$

So both series behave the same

Since $\sum \frac{(-1)^n}{\sqrt{n}}$ is Conditionally Convergent, so is $\sum \frac{(-1)^n(n^3+1)}{\sqrt{n^7+2}}$

(c)

$$\sum_{n=0}^{\infty} \frac{3^{2n}}{2^{3n}} = \sum_{n=0}^{\infty} \frac{9^n}{8^n}$$ $\to$ Geometric with $r = \frac{9}{8}$

Since $r > 1$, our series Diverges.

(d)

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$$

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{2n!}{n!}$

$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$

So $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$ Converges Absolutely

4. (a) Determine a series for $f(x) = \cos(x^3)$.

$$f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots$$

(b) Is $f(x) = \cos(x^3)$ an even function or an odd function? How do you know?

Cos(x^3) is even, as only even powers of x appear in the expansion.

(c) Find the *twelfth* derivative of $f(x) = \cos(x^3)$ evaluated at $x = 0$ (Leave your answer in terms of factorials).

$$f^{(12)}(0) = 12! \cdot c_{12}$$

$$= 12! \cdot \frac{1}{4!} = \frac{12!}{4!}.$$

(d) Find the *thirteenth* derivative of $f(x) = \cos(x^3)$ evaluated at $x = 0$.

$$f^{(13)}(0) = 13!\, c_{13} = 13!\,(0) = 0.$$

2. Find the radius and interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n^2 2^n}.$$

Ratio Test: $\lim_{n\to\infty} \frac{(3x-2)^{n+1}}{(n+1)^2 2^{n+1}} \cdot \frac{n^2 2^n}{(3x-2)^n} = \lim_{n\to\infty} \frac{3x-2}{2} \frac{n^2}{(n+1)^2} =$

$\lim_{n\to\infty} \frac{3x-2}{2} \left(\frac{n}{n+1}\right)^2$

$= \frac{3x-2}{2}$

$\left|\frac{3x-2}{2}\right| < 1 \Rightarrow -1 < \frac{3x-2}{2} < 1$

$\Rightarrow -2 < 3x-2 < 2$

$\Rightarrow 0 \underset{?}{\leq} x \underset{?}{\leq} \frac{4}{3}$

$x = 0$; $\sum \frac{(-2)^n}{n^2 2^n} = \sum \frac{(-1)^n}{n^2}$ C

$x = \frac{4}{3}$ $\sum \frac{2^n}{n^2 2^n} = \sum \frac{1}{n^2} \to C$

Radius $= \frac{2}{3}$

Interval: $[0, \frac{4}{3}]$.

3. Calculate the following sums.

(a) $2 - \frac{3}{2} + \frac{9}{8} - \frac{27}{32} + \ldots$

Sum: Geometric

1st term $= 2$

ratio $= -3/4$

$S = \frac{2}{1 - \frac{-3}{4}} = \frac{2}{\frac{7}{4}} = \frac{8}{7}$.

(b) $\frac{\pi}{3} - \frac{\pi^3}{3^3 3!} + \frac{\pi^5}{3^5 5!} - \frac{\pi^7}{3^7 7!} + \ldots$

Series: $\sin(x)$ @ $x = \frac{\pi}{3}$

Sum $= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

7. Find the following indefinite integral via u-substitution two different ways:

$$\int \sin(x)\cos(x)\,dx$$

(a) By letting $u = \sin(x)$

$u = \sin x$

$du = \cos x$

$\Rightarrow \int u\,du = \frac{u^2}{2} + C$

$= \frac{\sin^2 x}{2} + \underline{C}$

(b) By letting $u = \cos(x)$.

$u = \cos x$

$du = -\sin x\,dx$

$\Rightarrow \int -u\,du = -\frac{u^2}{2} + C = -\frac{\cos^2 x}{2} + \underline{C}$

Explain why your two answers are the same, even though they look different.

Note $\frac{\sin^2 x}{2} + \frac{\cos^2 x}{2} = \frac{1}{2}$

so $\frac{\sin^2 x}{2} = \frac{1}{2} - \frac{\cos^2 x}{2}$

i.e. They Differ By a Constant

8. Suppose that $f(x)$ is a function that is both even and odd. Prove that $f(x) = 0$ for all x.

$f(x)$ even $\Rightarrow f(-x) = f(x)$

$f(x)$ odd $\Rightarrow f(-x) = -f(x)$

$\longrightarrow f(x) = -f(x)$

$2f(x) = 0 \Rightarrow f(x) = 0$

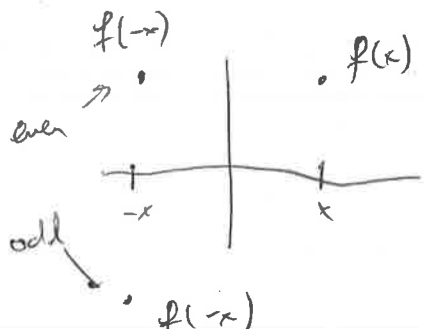
$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots$ OR

$f(x)$ even $\Rightarrow c_{2k+1} = 0$ for all taylor coeff's

$f(x)$ odd $\Rightarrow c_{2k} = 0$ for all Taylor coeff's

$c_n = 0$ for all $n \Rightarrow f(x) = 0 + 0x + 0x^2 + \ldots$

OR



undefined unless $f(x) = 0$ for all x

5. Find

$$\frac{d}{dx}\int_5^{x^2+x} \ln(\sqrt{t+1})\, dt.$$

FTC: $\frac{d}{dx}\left[F(x^2+x) - F(5)\right]$ where $F'(x) = \ln\sqrt{t+1}$

$= F'(x^2+x)(2x+1) - 0$

$= 2x+1 \ln\left(\sqrt{x^2+x+1}\right)$

6. Find the following antiderivatives and integrals.

(a) $\displaystyle\int \frac{x^3+1}{x^2}\, dx$ $= \displaystyle\int \frac{x^3}{x^2} + \frac{1}{x^2}\, dx = \int x + \frac{1}{x^2}\, dx$

$= \dfrac{x^2}{2} - \dfrac{1}{x} + C$

(b) $\displaystyle\int_0^1 xe^{x^2}\, dx$ $= \displaystyle\frac{1}{2}\int_{x=0}^{x=1} e^u\, du = \frac{1}{2} e^u \Big|_{x=0}^{x=1} = \frac{1}{2} e^{x^2}\Big|_{x=0}^{x=1}$

$u = x^2$

$du = 2x\, dx$

$= \frac{1}{2}(e-1)$

(c) $\displaystyle\int \tan(3x)\, dx$ $= \dfrac{-\ln|\cos(3x)|}{3} + C$