October 22, 2014

1a) $\sum_{n=1}^{\infty} \frac{\ln(n)}{2n+1} > \sum_{n=1}^{\infty} \frac{1}{2n+1} \sim \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges

comparison test

If $a_n \geq b_n$ and b_n diverges, a_n also diverges

limit comparison test

$\lim_{n\to\infty} \left(\frac{\frac{1}{2n+1}}{\frac{1}{n}} \right) = \frac{1}{2}$ Finite non-zero so series behave the same

b) $\sum_{n=0}^{\infty} \frac{2n+1}{\sqrt{3n^2+2}}$ $\lim_{n\to\infty} \frac{2n+1}{\sqrt{3n^2+2}} = \frac{2}{\sqrt{3}} \neq 0$ so series diverges ($\lim_{n\to\infty} a_n \neq 0$)

c) $\sum_{n=0}^{\infty} \left(\frac{2n+1}{4n-2}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ $L > 1$ D, $L < 1$ C, $L = 1$?

$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+1}{4n-2}\right)^n\right|} = \lim_{n\to\infty} \frac{2n+1}{4n-2} = \frac{1}{2}$

converges absolutely

d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+1}}$ Alternating series test

$\lim_{n\to\infty} \frac{1}{(n^2+1)^{\frac{1}{3}}} = 0$ so series converges somehow

Rewrite series w/o alternator

$\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^{\frac{1}{3}}}$ effective power is $n^{2/3}$ where $p<1$ so diverges

$\sim \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Since series diverges w/o alternator, this is conditionally convergent

e) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3}$ Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ $L<1$ AC, $L>1$ D, $L=1$?

$\lim_{n\to\infty} \left| \frac{(2(n+1))!}{((n+1)!)^3} \cdot \frac{(n!)^3}{(2n)!} \right| = \lim_{n\to\infty} \frac{(2n+2)!\; n!\, n!\, n!}{(2n)!\,(n+1)!\,(n+1)!\,(n+1)!} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^3} = 0$

Series is absolutely convergent

Remember that last step is considering limit of sequence, not series

f) $\sum_{n=1}^{\infty} \frac{\sin(n)}{\sqrt{n^3+1}}$ $(\leq)$ $\sum \frac{1}{\sqrt{n^3+1}} \sim \frac{1}{n^{3/2}}$ which converges because $p>1$

(because $\sin(n) \leq 1$)

absolutely convergent because abs value of series would also converge since $|\sin(n)| \leq 1$

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2. $f(x) = \sin(x^2)$ $\qquad$ $f(x) = \sin(x^3)$

even:

odd

$f(-x) = f(x)$ $\qquad$ $f(-x) = -f(x)$

$$f(x) = \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$ Look at powers of x

even powers $\Longrightarrow$ even function $\qquad$ $\sin((-x)^2) = \sin(x^2)$

$$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \dots$$ odd powers $\Longrightarrow$ odd function

$\sin((-x)^3) = \sin(-x^3) = -\sin(x^3)$

3a) $4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \dots$ Geometric series where $r = \frac{1}{3}$ $\quad a = 4$

$$\text{Sum} = \frac{a}{1-r} = \frac{4}{1-\frac{1}{3}} = \frac{4}{\frac{2}{3}} = 6$$

b) $$1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$= e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{n!}$$ where $x = -1$ so sum $= e^{-1} = \frac{1}{e}$

c) $$1 - \frac{\pi^2}{2} + \frac{\pi^4}{24} - \frac{\pi^6}{720} + \dots$$ sum $= \cos \pi = -1$

$$= \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$ where $x = \pi$

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4. For which x does $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{\sqrt{n+1}}$ converge?

Ratio test $$\lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(2x+3)^n} \right| = \lim_{n\to\infty} \frac{(2x+3)(\sqrt{n+1})}{\sqrt{n+2}}$$ — Goes to 1

$= 2x+3$ $\qquad$ For series to converge $|2x+3| < 1$

$-1 < 2x+3 < 1$

$-4 < 2x < -2$

$-2 < x < -1$ $\qquad$ Test endpoints

If $x = -2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ which converges (include -2)

If $x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{(1)^n}{\sqrt{n+1}}$ which diverges (p-series) (don't include -1)

$x = [-2, -1)$ $\qquad$ Radius of convergence $= \frac{1}{2}$

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5. $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$ (even function)

$$\lim_{x\to\infty} \frac{e^{x^2} - 1 - x^2 - \frac{x^4}{2}}{x^6} = \lim_{\to\infty} \frac{\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}\right) - 1 - x^2 - \frac{x^4}{2}}{x^6}$$

$$= \lim_{x\to\infty} \frac{x^6\left(\frac{1}{3!} + \frac{x^2}{4!} + \frac{x^4}{5!} + \dots\right)}{x^6} = \lim_{x\to\infty} \frac{1}{3!} + \frac{x^2}{4!} + \frac{x^4}{5!} = \boxed{\frac{1}{6}}$$

$f(x) = e^{x^2}$ $\quad f''(0) = 2 \quad f'''(0) = 0 \quad f''''(0) = \frac{4!}{2!}$

6. $$\frac{d}{dx}\int_{2x}^{3x} \frac{1}{\sqrt{1+t^2}}\,dt = \frac{d}{dx}\left(F(3x) - F(2x)\right) \text{ where } F'(t) = \frac{1}{\sqrt{1+t^2}}$$

Fundamental theorem of calculus

$$= 3F'(3x) - 2F'(2x) \quad \text{(chain rule)}$$

$$= 3\left(\frac{1}{\sqrt{1+(3x)^2}}\right) - 2\left(\frac{1}{\sqrt{1+(2x)^2}}\right)$$

7a) $$\int_{-2}^{2} (x+1)(x^2+1)\,dx = \int_{-2}^{2} x^3 + x^2 + x + 1\,dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-2}^{2} = \left[\frac{2^4}{4} + \frac{2^3}{3} + \frac{2^2}{2} + 2\right] - \left[\frac{2^4}{4} - \frac{2^3}{3} + \frac{2^2}{2} - 2\right]$$

$$= \frac{16}{3} + 4$$

b) $\int \cos^2(3x)\,dx$ $\qquad u = 3x$

$= \frac{1}{3}\int \cos^2(u)\,dx$ $\qquad du = 3\,dx$

$$= \frac{1}{3}\int \frac{1+\cos(2u)}{2}\,du = \frac{1}{3}\left[\frac{u}{2} + \frac{\sin(2u)}{4}\right] + c = \boxed{\frac{1}{3}\left[\frac{3x}{2} + \frac{\sin 6x}{4}\right] + c}$$

OR $$= \int \frac{1+\cos(6x)}{2}\,dx = \boxed{\frac{x}{2} + \frac{\sin(6x)}{12} + c}$$

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7c) $\int \frac{dx}{1+x}$ $u = 1+x$ $\int \frac{du}{u} = \ln|u| + c = \ln|1+x| + c$

$du = dx$

c') $\int \frac{dx}{1+x^2} = \arctan x + c$

Ex| $\int \frac{2x+1}{x^2+1} dx = \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx$

d) $\int \frac{\ln(2x+4)}{x+2} dx$ $u = \ln(2x+4)$

$du = \frac{1}{2x+4}(2)\,dx = \frac{1}{x+2} dx$

$\int u\,du = \frac{u^2}{2} + c = \frac{(\ln(2x+4))^2}{2} + c$

OR $= \int \frac{\ln|2(x+2)|}{x+2} dx = \int \frac{\ln|2| + \ln|x+2|}{x+2} dx$

Test Review

october 21, 2014

1. a) $\sum_{n=1}^{\infty} \frac{\ln n}{2n+1} \geq \sum_{n=1}^{\infty} \frac{1}{2n+1} \sim \sum_{n=1}^{\infty} \frac{1}{n}$

(first ↑: comparison test; second ↑: limit comparison)

$\sum_{n=1}^{\infty} \frac{1}{2n+1}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ behave the same; both diverge. series is **Divergent**

1. b) $\sum_{n=0}^{\infty} \frac{2n+1}{\sqrt{3n^2+2}} \rightarrow \lim_{n\to\infty} \frac{2n+1}{\sqrt{3n^2+2}} = \frac{2}{\sqrt{3}} \rightarrow$ **Diverges** b/c values don't go to 0.

1. c) $\sum_{n=0}^{\infty} \left(\frac{2n+1}{4n-2}\right)^n \rightarrow$ ROOT TEST $\rightarrow \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+1}{4n-2}\right)^n} \rightarrow \lim_{n\to\infty} \frac{2n+1}{4n-2} = \frac{1}{2} < 1$

series is **absolutely convergent.** b/c $\frac{1}{2} < 1$.

1. d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+1}} \rightarrow$ Alternating Series Test $\rightarrow \lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2+1}} = 0 \rightarrow$ some convergence.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}} \rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{2/3}} \rightarrow$ Divergent b/c $p < 1$ $\left(\frac{1}{n^p}\right)$ **Conditionally Convergent**

1. e) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3} \rightarrow$ RATIO TEST $\rightarrow \lim_{n\to\infty} \left| \frac{2(n+1)!}{((n+1)!)^3} \cdot \frac{(n!)^3}{(2n)!} \right| \rightarrow \lim_{n\to\infty} \frac{(2n+2)!\ n!\ n!\ n!}{(2n!)(n+1)!(n+1)!(n+1)!} \rightarrow$

$\lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^3} = 0 \rightarrow$ series is **Absolutely Convergent**

TEST REVIEW

October 22, 2014

1. f) $\sum_{n=1}^{\infty} \frac{\sin(n)}{\sqrt{n^3+1}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} \sim \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ → P series greater than 1.

Series is **absolutely convergent**

2. $f(x) = \sin(x^2)$ and $f(x) = \sin(x^3)$

even function: $f(-x) = f(x)$

odd function: $f(-x) = -f(x)$

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots$ **even function**

$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \dots$ **odd function**

3. a) $4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} \dots$ (each term $\times \frac{1}{3}$)

Sum $= \frac{a}{1-r} \rightarrow \frac{4}{1-\frac{1}{3}} = \frac{4}{\frac{2}{3}} = \boxed{6}$

3. b) $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} \dots$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \rightarrow \boxed{\frac{1}{e}}$

3. c) $1 - \frac{\pi^2}{2} + \frac{\pi^4}{24} - \frac{\pi^6}{720} + \dots$ $\quad \cos(\pi) = \boxed{-1}$

4. For which x does $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{\sqrt{n+1}}$ converge? → $\lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(2x+3)^n} \right|$ →

$\lim_{n\to\infty} \left| (2x+3) \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} \right|$ (→ goes to 1.) → $|2x+3| \overset{?}{\leq} 1$ → $-1 \overset{?}{\leq} 2x+3 \overset{?}{\leq} 1$ →

$-2 \overset{?}{\leq} x \overset{?}{\leq} -1$ → $x=-2$: $\sum \frac{(-4+3)^n}{\sqrt{n+1}}$ → converge!

$x=-1$: $\sum \frac{(-2+3)^n}{\sqrt{n+1}}$ → Diverges!

$\boxed{-2 \leq x < -1 \quad x \in [-2, -1)}$

TEST REVIEW

October 28, 2015

1. a) $\sum_{n=1}^{\infty} \frac{\sin(n)}{4n^2+1} \le \sum_{n=1}^{\infty} \frac{1}{4n^2+1} \sim \sum_{n=1}^{\infty} \frac{1}{4n^2}$ → p series greater than 1

Series is **absolutely convergent**

2. $f(x) = \sin(x^2)$ and $f(x) = \sin(x^3)$

even function: $f(-x) = f(x)$

odd function: $f(-x) = -f(x)$

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots$ **even function**

$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} \dots$ **odd function**

3. a) $4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} \dots$ ($\times \frac{1}{3}$) sums $\frac{a}{1-r}$ → $\frac{4}{1-\frac{1}{3}} = \frac{4}{\frac{2}{3}} = \boxed{6}$

b) $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} \dots$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ → $\boxed{\frac{1}{e}}$

c) $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots$ $\cos(\pi) = \boxed{-1}$

4. For which x does $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{\sqrt{n+1}}$ converge? → $\lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(2x+3)^n} \right|$ →

$\lim_{n\to\infty} \left| (2x+3) \frac{\sqrt{n+1}}{\sqrt{n+2}} \right|$ (goes to 1) → $|2x+3| \le 1$ → $-1 \le 2x+3 \le 1$ →

$-2 \le x \le -1$ → $x=-2$: $\sum \frac{(-1)^n}{\sqrt{n+1}}$ → converges

$x=-1$: $\sum \frac{(1)^n}{\sqrt{n+1}}$ → diverges

$\boxed{-2 \le x < -1 \quad x \in [-2, -1)}$

TEST REVIEW

October 22, 2014

5. $f(x) = e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$

$\lim_{x\to 0} \frac{e^{x^2} - 1 - x^2 - \frac{x^4}{2}}{x^6} \to \lim_{x\to 0} \frac{(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} \ldots) - 1 - x^2 - \frac{x^4}{2!}}{x^6} \to$

$\lim_{x\to 0} \frac{1}{3!} + \frac{x^2}{4!} + \frac{x^4}{5!} \cdots = \boxed{\frac{1}{6}}$

6. $\frac{d}{dx} \int_{2x}^{3x} \frac{1}{\sqrt{1+t^2}}\, dt \to$ FTC $\to \frac{d}{dx}\left(F(3x) - F(2x)\right)$ where $F'(t) = \frac{1}{\sqrt{1+t^2}} \to$

$\boxed{\frac{3}{\sqrt{1+(3x)^2}} - \frac{2}{\sqrt{1+(2x)^2}}}$

7. a) $\int_{-2}^{2} (x+1)(x^2+1)\, dx \to \int_{-2}^{2} (x^3 + x^2 + x + 1)\, dx \to \left[\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-2}^{2} \to$

$\left[\frac{2^4}{4} + \frac{2^3}{3} + \frac{2^2}{2} + 2\right] - \left[\frac{(-2)^4}{4} + \frac{(-2)^3}{3} + \frac{(-2)^2}{2} - 2\right] = \frac{16}{3} + 4 = \boxed{\frac{28}{3}}$

7. b) $\int \cos^2(3x)\, dx \to \int \frac{1 + \cos 6x}{2}\, dx \to \boxed{\frac{x}{2} + \frac{\sin 6x}{12} + C}$

7. c) $\int \frac{dx}{1+x} \to \boxed{\ln|1+x| + C}$

7. d) $\int \frac{\ln(2x+4)}{x+2}\, dx \to u = \ln(2x+4)$, $du = \frac{1}{2x+4} \cdot 2\,dx \to du = \frac{1}{x+2}\, dx \to$

$\int u\, du \to \frac{u^2}{2} + C \to \boxed{\frac{[\ln(2x+4)]^2}{2} + C}$