KEY

Math 126: Midterm Exam
October 23, 2015

You have 90 minutes to complete this closed-book, closed-notes, closed-colléague exam. You may use a calculator for arithmetic only (i.e., trigonometry, exponential, and logarithmic functions, but no Calculus and no Graphing). Please sign the following statement when you've finished.

> I certify that the work on this exam is entirely my own. I have not used a calculator for purposes other than arithmetic. I have not discussed the specific contents of this exam with anyone else prior to my taking it.
>
> *Signature:* Mr. Key

1. Tell whether each series converges conditionally, converges absolutely, or diverges. Be sure to use an appropriate test to justify your answers.

   (a)
   $$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^3+n}}$$

   The series is ALTERNATING, $\lim_{n\to\infty} \frac{1}{\sqrt{n^3+n}} = 0$

   and $\frac{1}{\sqrt{(n+1)^3+(n+1)}} > \frac{1}{\sqrt{n^3+n}}$

   So the series <u>Converges</u>

(b)

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{4^n}$$

$$\underbrace{\sum_{n=0}^{\infty} \frac{2^n}{4^n}}_{C} + \underbrace{\sum_{n=0}^{\infty} \frac{3^n}{4^n}}_{C} = \frac{1}{1-\frac{1}{2}} + \frac{1}{1-3/4} = 2+4=6$$

↳ Converges!

(c)

$$\sum_{n=0}^{\infty} \frac{\sqrt{n^2+1}}{n^2+3}$$

We use Lim. Comparison

$$\frac{\sqrt{n^2+1}}{n^2+3} \sim \underbrace{\frac{1}{n}}_{\text{Divergent!}} \qquad \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n^2+3} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\sqrt{n^4+n^2}}{n^2+3} \to 1$$

So the series behave the same

Thus ... both Diverge

(d)

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{3^n (n!)}$$

Ratio test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{\cancel{1 \cdot 3 \cdots (2n+1)}(2n+3)}{3^{n+1}(n+1)!} \cdot \frac{3^n\, n!}{\cancel{1 \cdot 3 \cdot 5 \cdots (2n+1)}\cancel{(2n+3)}}$

$$= \frac{2n+3}{3(n+1)} \to 2/3$$

so ratio < 1

$\Rightarrow$ Absolute Convergence

2. Find the radius and interval (with endpoints!) of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(5-x)^n}{2^n\sqrt{n}}.$$

Ratio test

$$\lim_{n\to\infty} \frac{(5-x)^{n+1}}{2^{n+1}\sqrt{n+1}} \cdot \frac{2^n\sqrt{n}}{(5-x)^n} = \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}^{\,1}}{\sqrt{n+1}} \, \frac{(5-x)}{2}$$

So we need $\left|\frac{(5-x)}{2}\right| < 1 \qquad -1 < \frac{5-x}{2} < 1$

$$\left.\begin{aligned} -2 &< 5-x < 2 \\ -7 &< -x < -3 \\ \Rightarrow\ 7 &> x > 3 \end{aligned}\right\} \Rightarrow \text{Int. of Conv } (3, 7]$$

Radius $= 2$.

$x=7$: $\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Converges

$x=3$: $\sum_{n=1}^{\infty} \frac{(2^n)}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ Diverges

3. Calculate the following sums.

(a) $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots$

Geometric, ratio $-\frac{2}{3}$

$$\Rightarrow S = \frac{2}{1-(-\frac{2}{3})} = \frac{2}{5/3} = \underline{6/5}$$

(b) $1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{3^n}{n!}$

$$= e^x\Big|_{x=3} = \underline{e^3}$$

(c) $1 + \frac{\pi}{3} - \frac{\pi^2}{3^2 2!} - \frac{\pi^3}{3^3 3!} + \frac{\pi^4}{3^4 4!} + \frac{\pi^5}{3^5 5!} - \ldots$

$$\left.\begin{aligned} &\text{odd terms} \to \sin(x)\Big|_{x=\pi/3} \\ &\text{even terms (start @ 0)} \to \cos x\Big|_{x=\pi/3} \end{aligned}\right\} \text{Sum} = \sin\frac{\pi}{3} + \cos\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}.$$

4. (a) What is the Taylor series (about $x = 0$) for $f(x) = \sin(x)$. Write the first few terms as well as a closed form in Σ-notation.

$$f(x) = \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(b) Use the series for $\sin(x)$ to give a series for $\sin(x^2)$. Write the first few terms as well as a closed form in Σ-notation.

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

(c) Integrating term-by-term, give a series for $\int \sin(x^2)\, dx$. Write the first few terms as well as a closed form in Σ-notation.

$$\int \sin(x^2)\,dx = \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$$

(d) Find

$$\frac{d}{dx}\int_{\cos(x)}^{\pi} e^{t^2}\,dt.$$

$$= \frac{d}{dx}\left[F(\pi) - F(\cos(x))\right] \text{ where } F'(t) = e^{t^2}$$

$$= 0 - (-\sin x)\,F'(\cos(x)) = \sin x\, e^{(\cos^2 x)}$$

5. Find the following antiderivatives and integrals.

(a) $\displaystyle\int \frac{\sin(x)+\cos(x)}{\cos(x)}\,dx = \int \frac{\sin x}{\cos x} + 1\,dx$

$$= \int \tan x + 1\,dx = -\ln|\cos x| + x + C$$

(b) $\displaystyle\int_0^1 e^x e^{e^x}\,dx \longrightarrow \int_1^e e^u\,du = e^u\Big|_1^e$

$u = e^x$ $\quad$ u.l. $= e^1 = e$

$du = e^x dx$ $\quad$ l.l. $= e^0 = 1$

$$= e^e - e.$$

(c) $\displaystyle\int x(x+1)^2\,dx \Rightarrow \int (u-1)\,u^2\,du$

let $u = x+1$ $\quad = \int u^3 - u^2\,du$

$du = dx$

$x = u-1$ $\quad = \dfrac{u^4}{4} - \dfrac{u^3}{3} + C = \dfrac{(x+1)^4}{4} - \dfrac{(x+1)^3}{3} + C$

6. Suppose that we know that

$$\int \tan^3(x)\,dx = \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C$$

Determine

$$\int \cot^3(x)\,dx$$

take the negative cofunctions

By the "metarule"

$\int \cot^3(x)\,dx = -\frac{1}{2}\cot^2 x - \ln|\sin(x)| + C$

7. Let $f(x)$ be an odd function and let $F(x)$ be an antiderivative of $f(x)$. Is F necessarily an even function? Explain.

$f(x) = c_1 x + c_3 x^3 + c_5 x^5 + \cdots$

$F(x) = C + c_1 \frac{x^2}{2} + c_3 \frac{x^4}{4} + c_5 \frac{x^6}{6} + \cdots$ ← even!

Yes.

8. Let $f(x)$ be an even function and let $F(x)$ be an antiderivative of $f(x)$. Is F necessarily an odd function? Explain.

$f(x) = c_0 + c_2 x^2 + c_4 x^4 + c_6 x^6 + \cdots$

$F(x) = C + c_0 x + c_2 \frac{x^3}{3} + c_4 \frac{x^5}{5} + \cdots$

not even because of the C.

No!