1. Evaluate

$$\lim_{n\to\infty} \sqrt{n^2+6n} - \sqrt{n^2+2n}$$

Preliminary numeric investigation suggests the form is $\infty - \infty$, indeterminate.

Plugging in larger numbers suggests the limit is 2. We verify algebraically

$$\lim_{n\to\infty} \sqrt{n^2+6n} - \sqrt{n^2+2n}\left(\frac{\sqrt{n^2+6n}+\sqrt{n^2+2n}}{\sqrt{n^2+6n}+\sqrt{n^2+2n}}\right) = \lim_{n\to\infty} \frac{(n^2+6n)-(n^2+2n)}{\sqrt{n^2+6n}+\sqrt{n^2+2n}}$$

conjugate

$$= \lim_{n\to\infty} \frac{4n}{\sqrt{n^2+6n}+\sqrt{n^2+2n}}$$

← effective powers both 1, compare coeff's

$$= \cancel{\lim} \ \frac{4}{2} = \boxed{2}$$

2. Find a formula for a_n, where

$$\{a_n\}_{n=1}^{\infty} = \left\{1, \frac{4}{7}, \frac{9}{17}, \frac{16}{31}, \frac{25}{49}, \frac{36}{71}, \dots\right\}$$

We look for a pattern in numerators, this is easily seen to be n^2.

Each denominator is one less than twice the numerator, thus $2n^2-1$.

Our formula is $a_n = \dfrac{n^2}{2n^2-1}$

To take the limit, note that both powers are equal (2), thus we compare coefficients

$$\lim_{n\to\infty} \frac{n^2}{2n^2-1} = \frac{1}{2}.$$

3. Find $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n-2}} - \frac{1}{\sqrt{n}}$.

Writing out the first few terms suggests a telescoping series

$$S = \left(\frac{1}{\sqrt{3}} - \cancel{\frac{1}{\sqrt{5}}}\right) + \left(\frac{1}{\sqrt{4}} - \cancel{\frac{1}{\sqrt{6}}}\right) + \left(\cancel{\frac{1}{\sqrt{5}}} - \cancel{\frac{1}{\sqrt{7}}}\right) + \left(\cancel{\frac{1}{\sqrt{6}}} - \frac{1}{\sqrt{8}}\right) + \left(\cancel{\frac{1}{\sqrt{7}}} - \frac{1}{\sqrt{9}}\right) + \cdots$$

3 (cont.)

We note that all terms but the $\frac{1}{\sqrt{3}}$ and $\frac{1}{2}$ cancel; AND that the cancelled terms get smaller and smaller.

Formally, the kth partial sum

$$S_k = \frac{1}{\sqrt{3}} + \frac{1}{2} - \frac{1}{\sqrt{k-1}} - \frac{1}{\sqrt{k}}$$ which approaches $\frac{1}{\sqrt{3}} + \frac{1}{2}$ in the limit.

4. We seek $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$. Numeric investigations suggest this limit approaches $2.71828\ldots = e$.

We prove by setting $a_n = \left(1+\frac{1}{n}\right)^n$ and $b_n = \ln(a_n) = \ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right)$

We then take

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \xrightarrow{L'H} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\left(\frac{-1}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

Since $b_n \to 1$ and $b_n = \ln a_n$, $a_n = e^{b_n}$ or $a_n \to e^1 = e$.

5. a) Alice wins if she tosses Heads on her turn before Bob tosses Heads. Alice can win on any odd numbered toss. *

Alice wins with prob

$$\underset{AH}{\frac{1}{2}} + \underset{AT}{\frac{1}{2}} \cdot \underset{BT}{\frac{1}{2}} \cdot \underset{AH}{\frac{1}{2}} + \underset{AT}{\frac{1}{2}} \cdot \underset{BT}{\frac{1}{2}} \cdot \underset{AT}{\frac{1}{2}} \cdot \underset{BT}{\frac{1}{2}} \cdot \underset{AH}{\frac{1}{2}} + \ldots = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{\frac{1}{2}}{3/4} = \frac{2}{3}.$$

geometric, ratio $= \frac{1}{4}$

1st term $= \frac{1}{2}$

b) Same type scenario, but now Alice wins with prob $\frac{1}{4}$ on any given turn, and the turn "passes" w/probability $3/4$.

Her overall probability is

geom, $a = \frac{1}{4}$, $r = \frac{9}{16}$

$$\frac{1}{4} + \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{1}{4} + \frac{9}{64} + \frac{81}{1024} + \ldots = \frac{\frac{1}{4}}{1-\frac{9}{16}} = \frac{4}{7}.$$