Math 126: Practice Exam 2

1. Let $f(x) = \sqrt{x}e^x$. Determine the volume of the solid generated by revolving the area under $f(x)$ between $x = 1$ and $x = 5$ about the x axis.

   SOLUTION: $V_{slice} = \pi y^2\ dx$, so $V_{solid} = \int_1^5 \pi x e^{2x}\ dx$. Solve this using integration by parts, $u = x$ and $dv = e^{2x}\ dx$, so $du = dx$ and $v = \frac{1}{2}e^{2x}$. Then $V_{solid} = \frac{\pi}{2}xe^{2x}]_1^5 - \frac{\pi}{2}\int_1^5 e^{2x}\ dx$, integrating the second part gives $V_{solid} = [\frac{\pi}{2}xe^{2x} - \frac{\pi}{4}e^{2x}]_1^5 = (\frac{5\pi}{2}e^{10} - \frac{\pi}{4}e^{10} - [\frac{\pi}{2}e^2 - \frac{\pi}{4}e^2] = \frac{9\pi}{4}e^{10} - \frac{\pi}{4}e^2$

2. Prove the formula for the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, Area$=\pi ab$.

   SOLUTION: See example 2 on page 491 for the details on the integration. Note that *again* the double angle formula is used to integrate $\cos^2(x)$.

3. Determine
$$\int \cos^3(x)\ dx$$

   SOLUTION: Rewrite the integrand as $(1-\sin^2(x))\cos(x)$, then let $u = \sin(x), du = \cos(x)\ dx$. The antiderivative is $\sin(x) - \frac{1}{3}\sin(x) + C$.

4. Determine
$$\int \frac{2x+1}{x^3+x^2}\ dx$$

   SOLUTION: Factor the denominator as $x^2(x+1)$. Then write
$$\frac{2x+1}{x^3+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}.$$
   Solving for A, B, and C yields $A = 1, B = 1, C = -1$, so
$$\int \frac{2x+1}{x^3+x^2}\ dx = \int \frac{1}{x} + \frac{1}{x^2} + \frac{-1}{x+1}\ dx = \ln|x| - \frac{1}{x} - \ln|x-1| + C$$
$$= \ln|\frac{x}{x-1}| - \frac{1}{x} + C$$

5. Determine
$$\int x \arctan x\ dx$$

   SOLUTION: Integration by parts. Let $u = \arctan x$ and $dv = x\ dx$, then $du = \frac{1}{1+x^2}\ dx$ and $v = \frac{x^2}{2}$. Then
$$\int x \arctan x\ dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\ dx.$$
   We can long divide the new integral, to rewrite the integrand as $1 - \frac{1}{1+x^2}$, the antiderivative of which is $x - \arctan x$. So, the resulting antiderivative is
$$\frac{1}{2}x^2 \arctan x - \frac{x}{2} + \frac{1}{2}\arctan x$$

6. Give strategies for each of the following integrals. In each case, state the methods and any substitutions you would use.

   (a)
   $$\int \frac{x^2 + x}{x + 2}\, dx$$

   SOLUTION: Use long division (or numerator massage) to rewrite the integral as the sum of a polynomial in x and a constant times $\frac{1}{x+2}$. Then integrate normally, using a u substitution to integrate the fraction.

   (b)
   $$\int \frac{x^2}{(4 - x^2)^{3/2}}\, dx$$

   SOLUTION: Use a trig substitution, $x = 2\sin\theta$, making sure to express the numerator as $4\sin^2\theta$ and dx as $2\cos\theta\, d\theta$. Then let the trig identities work their magic. :)

   (c)
   $$\int \frac{\sec^6(2x)}{\tan^2(2x)}\, dx$$

   SOLUTION: First, substitute $u = 2x$ to avoid further substitution issues. Then rewrite the numerator as $\sec^4(u)\sec^2(u) = (1 + \tan^2(u))^2$, then make the substitution $w = \tan u$, and $dw = \sec^2 u\, du$. The integral is now a polynomial and is eminently solvable.

7. Determine
   $$\int_2^\infty \frac{1}{x^2 - 1}\, dx$$

   SOLUTION: First, observe that the function is defined and continuous everywhere on the interval, so there are no improprieties other than the infinite limit. We use partial fractions to get

   $$\lim_{b\to\infty} \int_2^b \frac{1}{x^2 - 1} = \lim_{b\to\infty} \frac{1}{2} \ln|\frac{x-1}{x+1}|\Big]_2^b = \frac{1}{2}(\ln 1 - \ln\frac{1}{3}) = \frac{\ln 3}{2}$$

   Note that L'Hôpital's Rule gives us the information on the limit and that the integral converges, as expected, since the integrand is approximately $\frac{1}{x^2}$

8. Use the comparison test to determine whether
   $$\int_3^\infty \frac{1}{x\cos x}\, dx$$

   converges or diverges.

   SOLUTION: We begin by observing that $x\cos x \leq x$, since $\cos x \leq 1$. Then $\frac{1}{x} \leq \frac{1}{x\cos x}$, and $\int_3^\infty \frac{1}{x}\, dx \leq \int_3^\infty \frac{1}{x\cos x}\, dx$ Since the integral on the left diverges, so does the integral on the right..
   Note that the integrals of the form $\int_a^\infty \frac{1}{x^p}\, dx$ are among the most useful for the comparison test.

9. Why is Simpson's Rule so much more accurate than other methods of approximating integrals?

   SOLUTION: Simpson's rule uses parabolas rather than rectangles or trapezoids to approximate the area under the curve. These parabolas encompass more area and have less wasted area then either the rectangles or the trapezoids. The accuracy is reflected in the maximum possible error given by Simpson's rule, which has a term of $\frac{1}{n^4}$, whereas the other rules have accuracy measures with terms of $\frac{1}{n^2}$.

10. Set up, but do not evaluate, the integrals necessary to determine the arc length and the surface area of the curve and solid in problem 1, and of the curve in problem 2.

    SOLUTION:

$$\text{ArcLength}_1 = \int_1^5 \sqrt{1 + (\frac{1}{2\sqrt{x}}e^x + \sqrt{x}e^x)^2}\ dx$$

$$\text{SurfaceArea}_1 = \int_1^5 2\pi\sqrt{x}e^x\sqrt{1 + (\frac{1}{2\sqrt{x}}e^x + \sqrt{x}e^x)^2}\ dx$$

$$\text{ArcLength}_1 = \int_0^a \sqrt{1 + (\frac{bx}{a\sqrt{a^2 - x^2}})^2}\ dx$$

    Note that these derivatives involve doing some mathematical calisthenics. Check them against your own answers and, as always, **SEE ME** if you have any questions.