KEY

Math 126: Midterm Exam

You have 90 minutes to complete this closed-book-closed-notes-closed-colleague exam. **READ ALL QUESTIONS AND DIRECTIONS CAREFULLY**. Please read the statement below and sign it when you are finished with the exam.

1. Determine if the following series converge or diverge, justify your answers with the appropriate tests. If the series is alternating, tell if it is absolutely or conditionally convergent.

18

(a)

$$\sum_{n=1}^{\infty} \frac{\ln n}{2n+1}$$

$\sum_{n=1}^{\infty} \frac{\ln n}{2n+1} \geq \sum_{n=1}^{\infty} \frac{1}{2n+1} \sim \sum_{n=1}^{\infty} \frac{1}{2n}$, so $\sum_{n=1}^{\infty} \frac{\ln n}{2n+1}$ is Divergent.

Divergent

LCT: $\frac{1}{2n+1} \cdot \frac{2n}{1} \to 1$ as $n \to \infty$

(b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+1}}$$

$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2+1}} = 0$, $\frac{1}{\sqrt[3]{n^2+1}} < \frac{1}{\sqrt[3]{(n+1)^2+1}}$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+1}}$ is Convergent,

but $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}} \sim \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ p-series, $p<1$, Divergent

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+1}}$ is Conditionally Convergent

(c)

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3}$$

Ratio test $a_n = \frac{(2n)!}{(n!)^3}$ $a_{n+1} = \frac{(2(n+1))!}{((n+1)!)^3}$

$R = \lim_{n\to\infty} \frac{(2n+2)!}{((n+1)!)^3} \cdot \frac{(n!)^3}{(2n)!} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^3} = \lim_{n\to\infty} \frac{4n^2+6n+2}{n^3+3n^2+3n+1}$

$\to 0$ so $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3}$ is Convergent!

12

2. For which values of x does the series

$$\sum_{n=1}^{\infty} \frac{(2x+3)^n}{\sqrt{n+1}}$$

converge? Be sure to check endpoints.

Ratio Test: $\lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(2x+3)^n} \right|$

$= \lim_{n\to\infty} \left| \sqrt{\frac{n+1}{n+2}} \cdot (2x+3) \right| = |2x+3| < 1$

$-1 < 2x+3 < 1$; $-4 < 2x < -2$

$-2 < x < -1$

Endpoints $x = -2 \to \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \to C$ (terms decrease to 0)

$x = -1 \to \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \to D$ (p-series, $p = \frac{1}{2}$)

Thus $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{\sqrt{n+1}}$ converges for $x \in [-2, -1)$ or $-2 \le x < 1$

12

3. Determine a series for $f(x) = e^{x^2}$ and use your series to determine

$$\lim_{x\to 0} \frac{e^{x^2} - 1 - x^2 - \frac{x^4}{2}}{x^6}$$

$f(x) = e^{x^2}$; $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

so $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$

$\lim_{x\to 0} \frac{e^{x^2} - 1 - x^2 - \frac{x^4}{2}}{x^6} = \lim_{x\to 0} \frac{\left(\cancel{1} + \cancel{x^2} + \cancel{\frac{x^4}{2!}} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right) \cancel{-1} \cancel{-x^2} \cancel{-\frac{x^4}{2}}}{x^6}$

$= \lim_{x\to 0} \frac{1}{3!} + \frac{x^2}{4!} + \frac{x^4}{5!} + \cdots = \frac{1}{6}$

4. Find

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\cos\left(\frac{\pi}{4}+\frac{\pi i}{4n}\right)$$

10

$\frac{b-a}{n} = \frac{\pi}{4n}$ $\quad b-a = \frac{\pi}{4}$ $\quad a = \frac{\pi}{4}$

$b = \pi/2$

$f(x) = \cos x$

$$\int_{\pi/4}^{\pi/2} \cos x\,dx = \sin x \Big|_{\pi/4}^{\pi/2}$$

$$\sin \pi/2 - \sin \pi/4 = 1 - \frac{\sqrt{2}}{2}$$

5. Find

$$\frac{d}{dx}\int_{2x}^{3x}\frac{1}{\sqrt{1+t^2}}\,dt$$

10

$$\frac{d}{dx}\int_{2x}^{3x}\frac{1}{\sqrt{1+t^2}}\,dt = \frac{d}{dx}\left(F(3x) - F(2x)\right) \quad \text{where } F'(t) = \frac{1}{\sqrt{1+t^2}}$$

$$= F'(3x)\cdot 3 - F'(2x)\cdot 2$$

$$= \frac{1}{\sqrt{1+9x^2}}\cdot 3 - \frac{1}{\sqrt{1+4x^2}}\cdot 2$$

6. Find the following integrals and antiderivatives:

18

(a)

$$\int_{-2}^{2} (x+1)(x^2+1)\,dx$$

$$\int_{-2}^{2} (1+x)(1+x^2)\,dx = \int_{-2}^{2} 1 + x + x^2 + x^3\,dx$$

$$= \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \right]_{-2}^{2} = 2 + \cancel{\frac{4}{2}} + \frac{8}{3} + \cancel{\frac{16}{4}} - \left[-2 + \cancel{\frac{4}{2}} - \frac{8}{3} + \cancel{\frac{16}{4}} \right]$$

$$= 4 + \frac{16}{3} = \frac{28}{3}$$

(b)

$$\int \frac{x}{x+2}\,dx$$

$$\int \frac{x}{x+2}\,dx$$

$u = x+2$

$du = dx$

$x = u-2$

$$\int \frac{u-2}{u}\,du = \int 1 - \frac{2}{u}\,du = u - 2\ln|u| + C$$

$$= x+2 - 2\ln|x+2| + C$$

(c)

$$\int \frac{1}{1+4x^2}\,dx$$

$u = 2x$

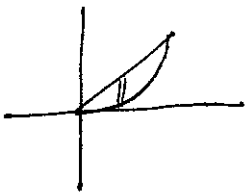
$du = 2dx$

$$= \frac{1}{2} \int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2} \arctan u = \frac{1}{2} \arctan(2x) + C$$

20

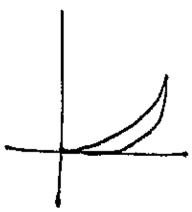
7. (a) Find the area between $y = x$ and $y = x^2$ in the first quadrant.



$$\int_0^1 x - x^2\,dx = \cancel{\tfrac{1}{6}}$$

$$\frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(b) Find the area between $y = x^2$ and $y = x^3$ in the first quadrant.



$$\int_0^1 x^2 - x^3\,dx$$

$$= \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

(c) Find the area between $y = x^3$ and $y = x^4$ in the first quadrant.

$$\int_0^1 x^3 - x^4\,dx = \frac{x^4}{4} - \frac{x^5}{5} = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

(d) Find the area between $y = x^4$ and $y = x^5$ in the first quadrant. (You need not explicitly integrate here).

Based on pattern:

$$\frac{1}{5} - \frac{1}{6} = \frac{1}{30}$$

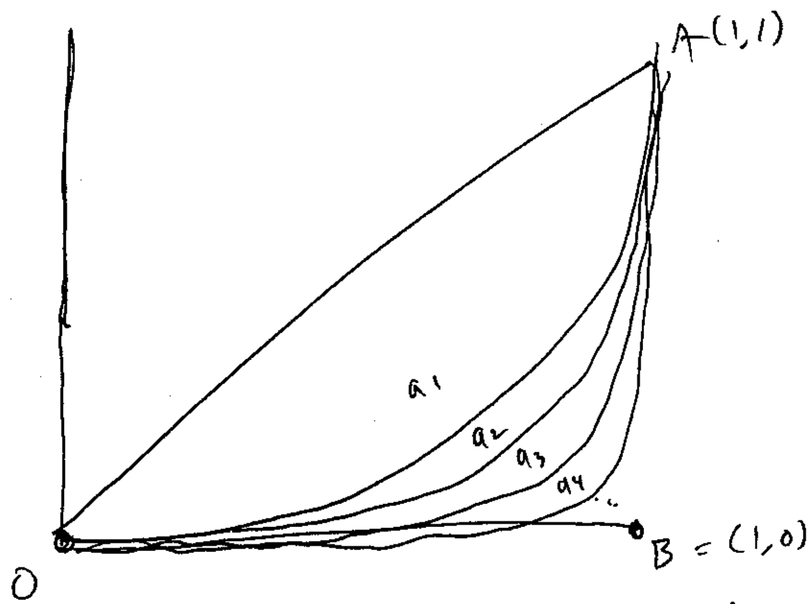
(e) Let a_n be the area between the graphs of $y = x^n$ and $y = x^{n+1}$ in the first quadrant. Find $\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} a_n = \left(\frac{1}{6}\right) + \left(\frac{1}{12}\right) + \left(\frac{1}{20}\right) + \left(\frac{1}{30}\right)$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \frac{1}{6}\right) + \dots$$

$$= \frac{1}{2}$$

(f) Draw the graphs of $y = x, y = x^2, y = x^3, y = x^4$, etc. on the same set of axes, and give a geometric justification to your answer in part (e).



$a_1 + a_2 + a_3 + a_4 + \dots$ = Area of $\triangle$ OAB $= \frac{1}{2}(1)(1) = \frac{1}{2}$