Practice Exam Key 1 Math 126 Spring 2010

1a) $\sum_{n=1}^{\infty} \frac{4^n}{5^n+n^5} \sim \sum_{n=1}^{\infty} \frac{4^n}{5^n}$ $\left(\lim_{n\to\infty} \frac{\frac{4^n}{5^n+n^5}}{\frac{4^n}{5^n}}\right)$

$$= \lim_{n\to\infty} \frac{4^n \cdot 5^n}{\underbrace{5^n \cdot 4^n}_{\text{dominant term}} + 4^n \cdot n^5} = 1$$

So both series behave the same.

Since $\sum_{n=1}^{\infty} \frac{4^n}{5^n}$ is geometric, with $r = \frac{4}{5} < 1$, $\sum_{n=1}^{\infty} \frac{4^n}{5^n}$ converges, so the series (a) is convergent

b) $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{2n^3+1}$ Since the denominator dominates, the terms go to 0, and so the series is some kind of convergent

Checking for absolute convergence

$$\sum_{n=1}^{\infty} \frac{(n+1)}{(2n^3+1)} \sim \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (\text{limit of the ratio} = \tfrac{1}{2})$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, The series (b) is Absolutely Convergent

c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ $\cos(n\pi) = (-1)^n$, so $(c) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is alternating harmonic, and this Conditionally convergent

d) $\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$ Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \left|\frac{(n+1)^2 2^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2 2^n}\right| = \lim_{n\to\infty} \left|\left(\frac{n+1}{n}\right)^2 \cdot \frac{2}{n+1}\right|$$

$$= \lim_{n\to\infty} \left|\frac{2}{n+1}\right| = 0 < 1$$

so (d) is Convergent

2) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}(1-x)^n$ Ratio test $\displaystyle\lim_{n\to\infty} \left| \frac{n+1}{(n+1)^2+1}(1-x)^{n+1} \cdot \frac{n^2+1}{n(1-x)^n} \right|$

$$= \lim_{n\to\infty} \left| \underbrace{\left(\frac{n+1}{n}\right)}_{1} \underbrace{\left(\frac{n^2+1}{n^2+2n+2}\right)}_{1} (1-x) \right| = |1-x| < 1$$

$\cancel{-1 < 1-x < 1}$ $\qquad -1 < 1-x < 1$

$\cancel{0 < -x < 2}$ $\qquad -2 < -x < 0$

$\cancel{-2 < x < 0}$ $\qquad 2 \underset{?}{>} x \underset{?}{>} 0$

Endpts $x = 2$

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}(1-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$$ Alternating, terms decreasing, Convergent

$$x=0 \quad \sum_{n=1}^{\infty} \frac{n}{n^2+1}(1-0)^n = \sum_{n=1}^{\infty} \frac{n}{n^2+1} \sim \sum_{n=1}^{\infty} \frac{1}{n} \text{ Divergent}$$

Interval of Convergence: $(0, 2]$

3) Series for $\sin(x^2)$

Easiest: $\displaystyle \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$

so $\displaystyle \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$

(replace $x \to x^2$)

4) $f(x) = x^3 + x$ about $x = 2$

	$x=2$	c_n
$f = x^3 + x$	10	$10/0! = 10$
$f' = 3x^2 + 1$	13	$13/1! = 13$
$f^{II} = 6x$	12	$12/2! = 6$
$f^{III} = 6$	6	$6/3! = 1$
$f^{IV} = 0$	0	0
$f^{V} = 0$	0	0
$f^{VI} = 0$	0	0

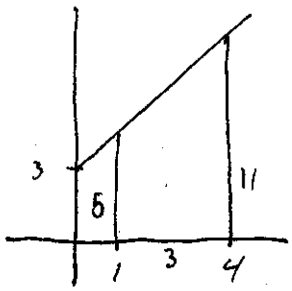
$f(x) = 10 + 13(x-2) + 6(x-2)^2 + (x-2)^3$

Expand:

$$\begin{aligned} &10 \\ &13x - 26 \\ 6x^2 &- 24x + 24 \\ x^3 - 6x^2 &+ 12x - 8 \\ \hline x^3 - 0x^2 &+ x - 0 \end{aligned}$$

$= x^3 + x$, which makes sense, since we're approximating a polynomial with a polynomial

5] $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{n=1}^{\infty} \underbrace{f\left(a+\frac{b-a}{n}i\right)}_{\text{height}}\underbrace{\left(\frac{b-a}{n}\right)}_{\text{width}}$

6] 

a) Geometry : Trapezoid $= h \cdot \frac{(b_1+b_2)}{2}$

$= 3\cdot\left(\frac{5+11}{2}\right) = 24$

b) $\int_1^4 2x+3\,dx = x^2+3x\Big|_1^4 = 16+12-(1+3)$

$= 28 - 4 = 24$

7) $\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{2}{n}\left(1+\frac{2i}{n}\right)^3$

$f\left(a+\frac{b-a}{n}i\right) = \left(1+\frac{2i}{n}\right)^3$

$\frac{b-a}{n} = \frac{2}{n}$, so $b-a=2$

$a=1$
$b=3$ $\quad f=x^3$

So $\sum = \int_1^3 x^3\,dx = \frac{x^4}{4}\Big|_1^3 = \frac{81}{4}-\frac{1}{4} = 20$

8) $\frac{d}{dx}\int_{x^2+x}^{\pi} \frac{\sin(t)}{t}\,dt$

$= \frac{d}{dx}\left[\underbrace{F(\pi)}_{\text{const.}} - F(x^2+x)\right]$ where $F'(t) = \frac{\sin(t)}{t}$

$= 0 - F'(x^2+x)\,(2x+1)$

$= -\left(\frac{\sin(x^2+x)}{x^2+x}\right)(2x+1)$

9) a) $\int \frac{x^3+1}{x^2}\,dx = \int \frac{x^3}{x^2}+\frac{1}{x^2}\,dx = \int x + x^{-2}\,dx = \frac{x^2}{2}+\frac{x^{-1}}{-1} = \frac{x^2}{2}-\frac{1}{x}+C$

b) $\int (x+3)\cos(x^2+6x+5)\,dx \rightarrow \frac{1}{2}\int \cos u\,du$

$u = x^2+6x+5$

$du = 2x+6\,dx$

$\frac{1}{2}\,du = (x+3)\,dx$

$= \frac{1}{2}\sin u + C$

$= \frac{1}{2}\sin(x^2+6x+5)+C$

c) $\int_0^a \frac{x}{\sqrt{1-x^2}}\,dx \rightarrow -\frac{1}{2}\int_1^{1-a^2}\frac{du}{\sqrt{u}}$

$u = 1-x^2$ $\quad$ u.l. $= 1-a^2$

$du = -2x$ $\quad$ l.l. $= 1-0^2 = 1$

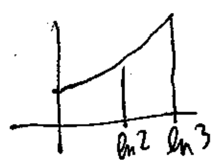
$= -\frac{1}{2}\,2u^{1/2}\Big]_1^{1-a^2}$

$= -\left[\sqrt{1-a^2}-1\right]$

$= 1-\sqrt{1-a^2}$

We need $|a|<1$, lest we either divide by 0 (if $a=\pm1$) or take the root of a negative.

10) $\int_0^{\ln 3} e^x\,dx = e^x\Big|_0^{\ln 3} = e^{\ln 3}-e^0 = 3-1 = 2$



half of area = 1

find c s.t.

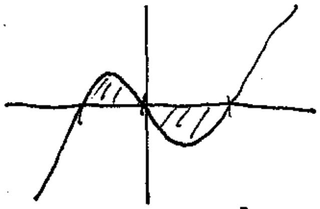
$\int_0^c e^x\,dx = 1 = e^c - e^0 = 1$ so $e^c = 2$, $c = \ln 2$

$\ln 2 = .6903 \rightarrow \frac{1}{2}\ln 3$

$\ln 3 = 1.0986$ ~~← 2 ln 3~~

So $c > \frac{1}{2}\ln 3$, which makes sense, since f is increasing

11. $y = x^3 - 9x$



Since $f(x)$ is odd, $\int_{-3}^{3} f(x) = 0$,

So $\int_{-3}^{0} f(x) = \int_0^3 f(x)$ $\boxed{\text{or the ratio of areas is } 1:1}$

12.

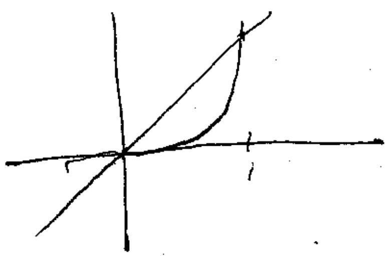
$\int_{-a}^{a} x^3 f(x)\,dx = 0$ for all $a \Rightarrow$

$x^3 f(x)$ is odd

That is $(-x)^3 f(-x) = -x^3 f(x)$

or that $f(-x) = f(x)$ $\boxed{\text{and } f \text{ must be } \underline{\text{even}}}$.

13 $\displaystyle\int_0^1 x - x^3\,dx$



$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{4}}$