

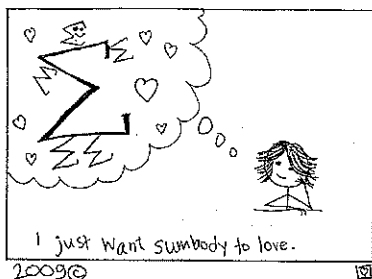
KEY

Math 126: Quiz 2

January 28, 2010

Feb. 11, 2010

You have the remainder of the hour to complete this closed-book, closed-notes, closed-colleague quiz. You may use a calculator for arithmetic only (ie, no plotting). PLEASE READ ALL DIRECTIONS CAREFULLY!



1. Short answer

- (a) Give an example of a series $\sum_{n=1}^{\infty} a_n$ whose terms approach 0, but which diverges.

The canonical example is $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series

- (b) Explain what it means for a series to be *conditionally convergent*.

A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

- (c) Explain the similarities in the Ratio Test and the Root Test.

Ratio Test $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ If $L < 1$, the series converges
Root Test $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ If $L > 1$ the series diverges
 $L = 1$ we can't tell

1 Both provide the same conditions as L for convergence!

2. For each series, tell whether it is convergent or divergent. For those convergent alternating series, determine whether the series absolutely converges or conditionally converges. Be sure to justify your answers by using the appropriate tests.

(a)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{\sqrt{n^5+1}} \quad b_n = \frac{n}{\sqrt{n^5+1}} \quad b_{n+1} = \frac{n+1}{\sqrt{(n+1)^5+1}}$$

4
Alternating Series $\lim_{n \rightarrow \infty} b_n = 0$ (compare exponents)

$b_{n+1} < b_n$ so converges

Compare $\sum b_n \sim \sum \frac{1}{n^{3/2}}$ $\lim_{n \rightarrow \infty} \frac{\frac{n}{\sqrt{n^5+1}}}{\frac{1}{n^{3/2}}} = \frac{n^{5/2}}{\sqrt{n^5+1}} \rightarrow 1$

Subsequences behave the same

Both Converge

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{\sqrt{n^5+1}}$ is Abs. Conv.

(b)

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$$

3
 $\sum_{n=1}^{\infty} \frac{\cos(n)}{2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n}$ Converges, geometric

$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$ is convergent

(c)

$$\sum_{n=1}^{\infty} \frac{4^n}{3^{2n}}$$

Ratio test $a_n = \frac{4^n}{3^{2n}} \quad a_{n+1} = \frac{4^{n+1}}{3^{2(n+1)}}$

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{3^{2n+2}} \cdot \frac{3^{2n}}{4^n} \right| = \frac{4}{9} < 1$

So convergent

(d)

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$$

3

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3} < \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

\boxed{C} \Leftarrow Convergent (p series)

(e)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \sin\left(\frac{1}{n}\right)$$

$$b_n = \sin\left(\frac{1}{n}\right)$$

$$\lim_{n \rightarrow \infty} b_n = 0$$

$$b_n > b_{n+1} \Rightarrow -C$$

Compare $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ with $\sum_{n=1}^{\infty} \frac{1}{n}$

4

$$\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 \text{ so both Diverge}$$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \sin\left(\frac{1}{n}\right)$ is conditionally
convergent

(f)

$$\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$$

Ratio test $a_n = \frac{n!(n+1)!}{(2n)!}$ $a_{n+1} = \frac{(n+1)!(n+2)!}{(2(n+1))!}$

4 $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(n+2)!}{(2n+2)!} \cdot \frac{(2n)!}{n!(n+1)!} \right|$
 $= \lim_{n \rightarrow \infty} \left| \frac{(n+2)(n+1)}{(2n+2)(2n+1)} \right| = \frac{1}{4} < 1$ so Convergent

3. (Bonus): Alice and Bob decide to play the game again. (Take turns tossing a fair coin, first head wins, Alice goes first.). This time, to make things fairer, Bob will get 2 tosses on each turn where Alice gets only one. Is the game now fair? Explain.

Alice still has the advantage. She will win 50% of the time on her first toss, so more than 50% of the time overall

$$P(A \text{ wins}) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \dots$$

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$$\frac{1}{2} + \frac{1}{16} + \frac{1}{128} + \dots = \frac{\frac{1}{2}}{1 - \frac{1}{8}} = \frac{\frac{1}{2}}{\frac{7}{8}} = \frac{8}{14} = \frac{4}{7}$$