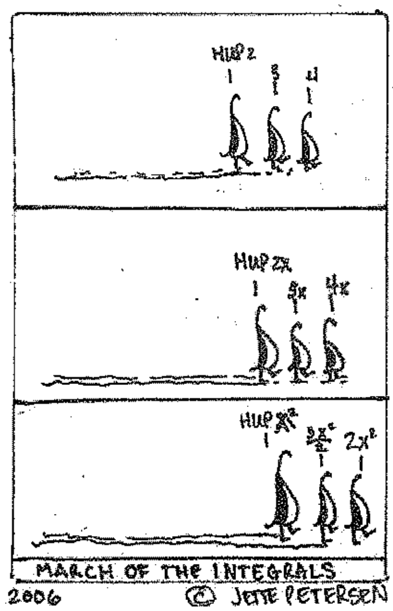
Key

Math 126: Quiz 3

February 25, 2010

You have the remainder of the hour to complete this closed-book, closed-notes, closed-colleague quiz. You may use a calculator for arithmetic only (ie, no plotting). PLEASE READ ALL DIRECTIONS CAREFULLY!



1. What is the difference between a Taylor series and a Maclaurin series?

A Maclaurin series is a Taylor series centered @ $x = 0$, but that is the only difference.

2. True or False:

If $f(x)$ is a continuous function and $a < b$, then $\int_a^b (f(x))^2 \, dx \geq 0$.

True. $(f(x))^2 \geq 0$, so $\int_a^b (f(x))^2 \, dx \geq 0$.

(any square is positive)

3. Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n3^n}$. Remember to check the endpoints.

$a_n = \frac{(x-1)^n}{n\,3^n}$ $\quad a_{n+1} = \frac{(x-1)^{n+1}}{(n+1)\,3^{n+1}}$ $\quad$ Ratio test: $\lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(x-1)^{n}}\right| = \left|\frac{x-1}{3}\right| < 1$; $-1 \overset{?}{\le} \frac{x-1}{3} \overset{?}{\le} 1$

$-3 \overset{?}{\le} x-1 \overset{?}{\le} 3$

$-2 \overset{?}{\le} x \overset{?}{\le} 4$

for $x = -2$, $a_n = \frac{(-1)^n}{n}$, series converges

for $x = 4$ $\quad a_n = \frac{1}{n}$, series diverges

Interval: $[-2, 4)$

4. (a) [1] Find a power series representation for $\frac{1}{1-x^3}$.

Geometric, ratio x^3, first term 1

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \dots = \sum_{n=0}^{\infty} x^{3n}$$

(b) Use part (a) to find a power series representation for $\frac{3x^2}{(1-x^3)^2}$.

$$\frac{d}{dx}\left(\frac{1}{1-x^3}\right) = -1(1-x^3)^{-2}\cdot(-3x^2) = \frac{3x^2}{(1-x^3)^2}$$

$$\text{Series: } 3x^2 + 6x^5 + 9x^8 + \dots = \sum_{n=0}^{\infty} (3n)\, x^{3n-1} = \frac{3x^2}{(1-x^3)^2}$$

(c) Use part (b) to find a power series representation for $\frac{1}{(1-x^3)^2}$.

factor out $3x^2$ from the answer to b ...

$$3x^2\left[1 + 2x^3 + 3x^6 + 4x^9 + \dots\right]$$

$$\hookrightarrow \frac{1}{(1-x^3)^2} = 1 + 2x^3 + 3x^6 + 4x^9 + \dots = \sum_{n=1}^{\infty} n\, x^{(3n-3)} = \sum_{n=1}^{\infty} n\, x^{3(n-1)}$$

[1] For full marks, write your answers in sigma notation. Barring this, write enough terms to establish a pattern

5. (a) Find the Taylor series expansion of $f(x) = e^x$ at $x = 1$.

	$x=1$	C_n
$f(x)=e^x$	e	$e/0!$
$f'(x)=e^x$	e	$e/1!$
$f''(x)=e^x$	e	$e/2!$
$f'''(x)=e^x$	e	$e/3!$

$$\longrightarrow \quad e^x = e + e(x-1) + e\frac{(x-1)^2}{2!} + e\frac{(x-2)^3}{3!} + \cdots$$

(note: $= e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right]$

$= e \cdot e^{x-1} = e^x$ ✓

(b) Take the derivative of your series. What do you notice?

$$\frac{d}{dx}\left[e + e(x-1) + e\frac{(x-1)^2}{2!} + e\frac{(x-1)^3}{3!} + \cdots\right]$$

$$= 0 + e + e(x-1) + e\frac{(x-1)^2}{2!} + \cdots$$

we get the same fcn back

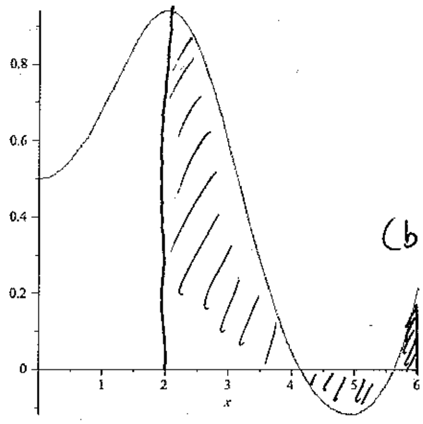
6. Determine the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{2n}(2n)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)\ \pi^{2n}}{(2n)!\, 2^{2n}} = 0$$

7. Consider the following graph for $f(x)$:



(b) < (c) < (d) < (a)

List the following integrals from least to greatest:

(a) $\int_2^{4.2} f(x)\,dx$ (b) $\int_{4.2}^{5.6} f(x)\,dx$ (c) $\int_2^{5.6} f(x)\,dx$ (d) $\int_2^{6} f(x)\,dx$

$b < 0$ $c = a + b < a$ $\int_{5.6}^{6} f(x)\,dx < -\int_{4.2}^{5.6} f(x)\,dx$

8. (Bonus) Alice and Bob are working on their Calc homework, and are asked to compute a series for $f(x) = \frac{1}{2-x}$. Alice figures

$$f(x) = \frac{1}{2-x} = \frac{1/2}{1-(x/2)} = (1/2) + (x/4) + (x^2/8) + (x^3/16) + \dots,$$

while Bob figures

$$f(x) = \frac{1}{2-x} = \frac{1}{1-(x-1)} = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \dots.$$

Who is correct? Explain.

Both are correct. Each has taken a power series at a different center. Alice's series converges for $-2 < x < 2$ while Bob's converges for $0 < x < 2$.