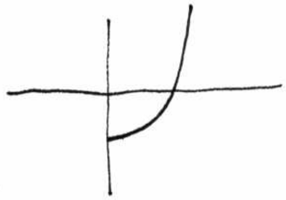
KEY

**Math 225: Exam the First**

You have 90 minutes to complete this test. This test is closed book, closed notes, and closed colleagues. You may use your calculator for the purposes of graphing equations, but be prepared to justify your answers algebraically.

1. (15 points) Let $x = t^2$ and $y = t^4 - 1$ be parametric equations.

   (a) Sketch the curve defined by these two equations.

   positive x values, or $x=0$

   y values $\geq 1$

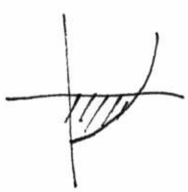
   (b) Determine $\frac{dy}{dx}$ and use it to determine the equation of the tangent line at $(4, 15)$.

   $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{2t} = 2t^2$$

   @ $(4,15)$ $t=2$ slope $=8$

   line: $y-15 = 8(t-4)$

   (c) Determine the area bound below by the curve and above and on the left by the axes.

   $$\int_a^b -y\,dx = \int_0^1 1-t^4(2t)\,dt = \int_0^1 2t-2t^5\,dt$$

   $$= t^2 - \frac{t^6}{3}\Big]_0^1 = \frac{2}{3}$$

   (d) Eliminate the parameter in the equations and verify your answers to parts *b* and *c*.

   $y = (t^2)^2 - 1 = x^2 - 1$

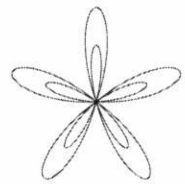
   $\frac{dy}{dx} = 2x = 2t^2$ ✓

   $y-15 = 8(x-4)$ ✓

   Area $= \int_0^1 1-x^2\,dx$

   $x - \frac{x^3}{3}\Big]_0^1 = \frac{2}{3}$ ✓

2. (15 points) Here is the plot of $r = 1 + 4\sin(5\theta)$.



(a) Describe the behavior of the graph in terms of the given equation. Specifically, explain maximum and minimum values, number of leaves, and the 'leaves within leaves'.

max $r = 5$
min $r = -3$ ← inner leaves, curve goes through origin & traces negative values.

$\sin 5\theta$ has 5 leaves, this graph has 5 leaves.

(b) Give an integral or integrals to determine the area outside a smaller leaf but inside a larger leaf. DO NOT ATTEMPT TO COMPUTE THIS INTEGRAL.

$r = 0$
$\sin(5\theta) = -1/4$

Large : $\int_{a}^{a+\pi/5} \frac{1}{2}(1+4\sin 5\theta)^2 \, d\theta$

Small : $\int_{\pi+a}^{\pi+a+\pi/5} \frac{1}{2}(1+4\sin(5\theta))^2 \, d\theta$

Area = Large − Small.

$a = \frac{1}{5}\arcsin(-1/4)$.

(c) How would changing the value of $a$ in the equation $r = 1 + a\,\cancel{\cos}\,\sin(5\theta)$ change the relative sizes of the inner and outer leaves? Focus on values $a \geq 1$. (Hint: How would we change the maximum and minimum values?)

As $a \to 1^+$, inner leaves shrink as min value $\to 0^-$

As $a \to \infty$, inner leaves approach outer leaves

as max value $= a+1$, min value $= -a+1$ } these get closer in absolute value.



The sizes of the leaves grow in absolute value as a increases.

3. (10 points) Calculate $\mathbf{i} \times \mathbf{j}$. Explain your result in terms of the $x, y$, and $z$ axes. What would $\mathbf{j} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{i}$ be?

$$\begin{array}{rl} \vec{i} & \langle 1,0,0\rangle \\ \times\ \vec{j} & \langle 0,1,0\rangle \\ \hline \end{array}$$

$\langle 0-0, 0-0, 1-0\rangle = \langle 0,0,1\rangle = \vec{k}$

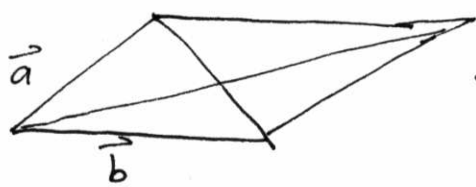
$\left.\begin{array}{l}\vec{i} \to x\text{-axis} \\ \vec{j} \to y\text{-axis}\end{array}\right\}$ perpendicular to both of these $\to z$-axis

$\underset{y}{\vec{j}} \times \underset{z}{\vec{k}} = \vec{i}$ $x$ axis

$\vec{k} \times \vec{i} = \vec{j}$.

4. (10 points)

(a) Draw a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$. Which vectors give the diagonals?




diagonals $\vec{a} + \vec{b}$ & $\vec{a} - \vec{b}$

(b) Prove: If the diagonals are orthogonal, then all sides of the parallelogram are equal.

Diagonals are orthogonal $\Rightarrow$

$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$

$\vec{a} \cdot \vec{a} + \cancel{\vec{b} \cdot \vec{a}} - \cancel{\vec{a} \cdot \vec{b}} - \vec{b} \cdot \vec{b} = 0$

$\|\vec{a}\|^2 - \|\vec{b}\|^2 = 0$ or $\|\vec{a}\|^2 = \|\vec{b}\|^2$

$\|\vec{a}\| = \|\vec{b}\|$, so all sides equal.

✓, ☺, etc.

5. (20 points) Consider the triangle formed by the points

$$A = (2, 1, -4), B = (-1, 1, -2), C = (3, 0, -3)$$

(a) Is $\angle B$ acute, right, or obtuse? Explain.

$\vec{BA} = \cancel{\langle 3, 0, 2 \rangle}\ \langle 3, 0, -2 \rangle$

$\vec{BC} = \langle 4, -1, -1 \rangle$

$\vec{BA} \cdot \vec{BC} = (12 + 0 + 2) = 14 > 0$ so $\measuredangle B$ is ACUTE

(b) What is the area of $\triangle ABC$?

Area $\triangle ABC = \frac{1}{2}$ Area Parallelogram $= \frac{1}{2} \| \vec{BA} \times \vec{BC} \|$

$$\begin{array}{r} \vec{BA} = \langle 3, 0, -2 \rangle \\ \times\ \vec{BC} = \langle 4, -1, -1 \rangle \\ \hline \langle 0-2, -8-{}^{-}3, -3-0 \rangle = \langle -2, -5, -3 \rangle \end{array}$$

$\frac{1}{2} \| \langle -2, -5, -3 \rangle \| = \frac{\sqrt{38}}{2}$.

(c) What is the equation of the plane that contains $\triangle ABC$?

$\vec{n} = \langle -2, -5, -3 \rangle$

point: $(2, 1, -4)$ (or others)

plane: $-2(x-2) - 5(y-1) - 3(z+4) = 0$

or $2x + 5y + 3z = -3$

6. (20 points) Below are a set of several equations in various 3 dimensional coordinate systems. Which of these equations represent the same surfaces? (By same, I mean the exact same surface in the same location). There are 9 equations, with three pairs and one triplet, each with the same surface. ALSO: Describe each surface in words.

(a) $x^2 - y^2 = z$

(b) $\rho = 2\cos(\phi)$

(c) $(x-1)^2 + y^2 = 1$

(d) $x = 0$

(e) $r^2 + z^2 = 2z$

(f) $r = 2\cos(\theta)$

(g) $\theta = \frac{\pi}{2}$

(h) $z = r^2\cos(2\theta)$

(i) $x^2 + y^2 + (z-1)^2 = 1$

a & h

$$\left.\begin{aligned} &x^2 - y^2 = z \\ &r^2\cos^2\theta - r^2\sin^2\theta = z \\ \text{Trig id} \to\ &r^2(\cos 2\theta) = z \end{aligned}\right\}$$ hyperbolic paraboloid (saddle)

b & e & i

$$\left.\begin{aligned} &\rho = 2\cos\phi \\ &\rho^2 = 2\rho\cos\phi \\ &x^2+y^2+z^2 = 2z \\ &r^2+z^2 = 2z \\ &x^2+y^2+(z-1)^2 = 2z \end{aligned}\right\}$$ sphere centered @ (0,0,1) of radius = 1 (sitting on top of xy plane)

c & f

$$\left.\begin{aligned} &(x-1)^2+y^2 = 1 \\ &x^2+y^2 = 2x \\ &r^2 = 2r\cos\theta \\ &r = 2\cos\theta \end{aligned}\right\}$$ cylinder of radius 1, axis parallel to z axis, offset by 1 unit along x axis.



d & g)

$$\left.\begin{aligned} &x = 0 \\ &r\cos\theta = 0 \\ &\theta = \pi/2 \end{aligned}\right\}$$ yz plane

7. (10 points) Do the following curves intersect? Do they collide? If so, where? Explain.

$$\mathbf{r}_1(t) = \langle \cos\left(\frac{\pi t}{4}\right), t+1, 2t-1 \rangle$$

$$\mathbf{r}_2(t) = \langle \sin\left(\frac{\pi s}{8}\right), \frac{s^2}{2}, \frac{s}{2} \rangle$$

(1) $\cos\left(\frac{\pi t}{4}\right) = \sin\left(\frac{\pi s}{8}\right)$

(2) $t+1 = \frac{s^2}{2}$

(3) $2t-1 = \frac{s}{2}$

either solve 2 & 3 simultaneously

$$\begin{array}{r} 2t+2 = s^2 \\ 2t-1 = \frac{s}{2} \\ \hline 3 = s^2 - \frac{s}{2} \end{array}$$

$2s^2 - s - 6 = 0$

$(2s+3)(s-2) = 0$

$s = \frac{-3}{2}$, $s = 2$

$t = \frac{7}{2}$

$t+1 = 4/2 = 2$, $t = 1$

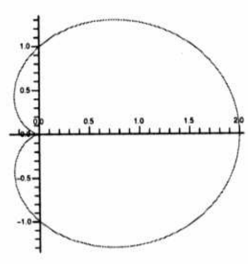
check w/(1), $\cos \pi/4 = \sin \pi/4$ ✓

$\cos\left(\frac{7\pi}{8}\right) \neq \sin\left(\frac{-3\pi}{16}\right)$

not $s = 3/2$, $t = 7/2$.

The curves intersect @ $\left(\frac{\sqrt{2}}{2}, 2, 1\right)$ but do not collide

8. Bonus: Spell the following (1 point each)



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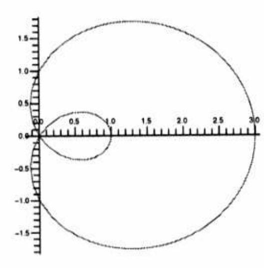

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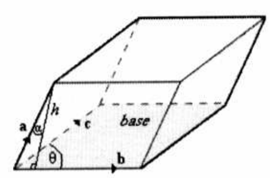


parallelepiped