KEY

Math 225: Exam the Second
November 8, 2006

You have 90 minutes to complete this closed-book, closed-notes, and closed-colleague exam. You may use a calculator but be prepared to justify your answers if you do so. **READ ALL QUESTIONS CAREFULLY**, as I am more lenient with partial credit if I feel you've done so.

1. Consider the function $f(x, y) = e^{x+y}$.

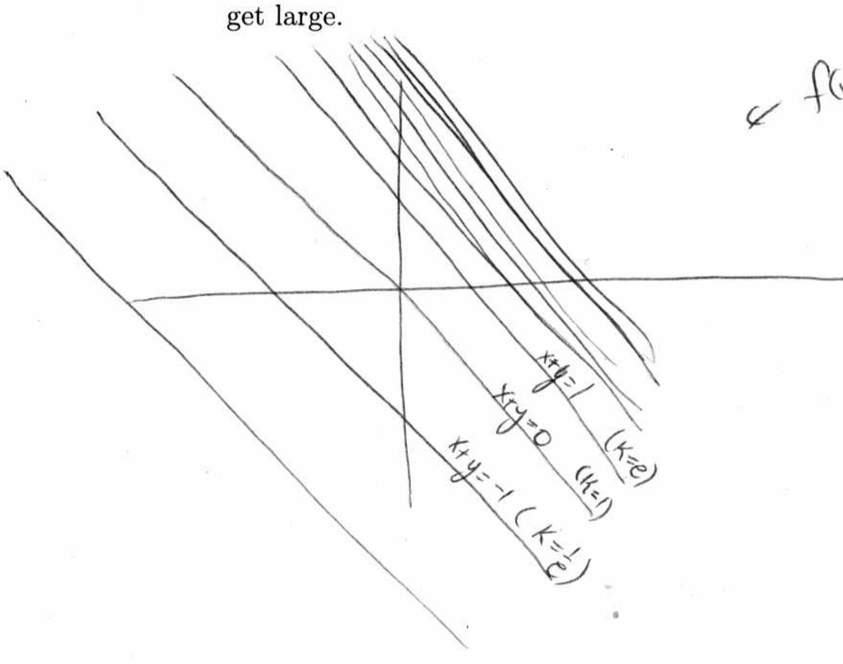
   (a) For which values k can one draw level curves of the form $f(x, y) = k$?

   $e^{x+y} > 0$, so for all values $k > 0$

   $e^{x+y} = k \Rightarrow x+y = \ln(k)$, so k must be greater than zero.

   (b) Draw enough of the level curves to describe the behavior of the function as x and y both get large.

   ← $f(x,y)$ gets steeper as x, y gets large.



2. Find

$$\lim_{(x,y)\to(1,1)} \frac{(x-y)}{\sqrt{x}-\sqrt{y}}$$

or explain why it doesn't exist.

Plugging in gives $\frac{1-1}{1-1} = \frac{0}{0}$, so further investigation required

$$\frac{x-y}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{(x-y)}} = \sqrt{x}+\sqrt{y}$$

$$\text{so } \lim_{(x,y)\to(1,1)} \frac{x-y}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(1,1)} \sqrt{x}+\sqrt{y} = 2.$$

3. Which of the following functions have $f_{xy} = 0$? Note: More than one answer is possible. Don't spend too much time on this one.

(a) $f(x,y) = x\cos(x^2) + y\arcsin(y^2)$
(b) $f(x,y) = y\cos(x^2) + x\arcsin(y^2)$
(c) $f(x,y) = x^{\cos(x)} - ye^{sin(y)}$
(d) $f(x,y) = \cos(x^2y^3)$

Both (a)&(c) have terms <u>only</u> in x or <u>only</u> in y.
(b)&(d) have mixed terms which imply that $f_x = g(x,y)$
$$\text{So } f_{xy} = \frac{\partial g}{\partial y} \neq 0.$$

So only (a) & (c) have $f_{xy} = 0$.

4. Let $f(x, y) = \sqrt{x^2 + y^3}$.

(a) Estimate $f(1.04, 1.98)$. near $f(1,2)$

$f(1,2) = \sqrt{1+8} = 3$

$f_x(1,2) = \dfrac{2x}{2\sqrt{x^2+y^3}} = \dfrac{x}{\sqrt{x^2+y^3}} \Rightarrow \dfrac{1}{3}$

$f_y(1,2) = \dfrac{3y^2}{2\sqrt{x^2+y^3}} = \dfrac{3y^2}{2\sqrt{x^2+y^3}} \Rightarrow \dfrac{12}{6} = 2$

$f(1.04, 1.98) \approx 3 + \frac{1}{3}(\cancel{1-1.04}) + 2(1.98-2)$
$1.04 - 1$

$= 3 + \frac{.04}{3} - .04 \quad = 3 - \frac{.08}{3} = 2.97333\ldots$

(b) Find the directional derivative of f at the point $(1, 2)$ in the direction of $\mathbf{i} + \mathbf{j}$

$u = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$D_{\vec{u}}(f) = \vec{\nabla} f \cdot \vec{u}$

$= \langle \frac{1}{3}, 2 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{1}{3\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{7}{3\sqrt{2}}$.

(c) Find the gradient and the maximum rate of change at the point (1,2).

$\vec{\nabla} f =$ gradient $= \langle \frac{1}{3}, 2 \rangle$

max rate of change $= \sqrt{\frac{1}{9} + 4} = \frac{\sqrt{37}}{3}$.

5. Let $f(x, y, z) = z(x^2 + y^2)$.

(a) Compute the partial derivatives f_x, f_y and f_z.

$f_x = 2xz$

$f_y = 2yz$

$f_z = x^2 + y^2$

(b) Convert the equation to cylindrical coordinates and compute f_r. Verify your answer using the chain rule.

$f(r\cos\theta, r\sin\theta, z) = r^2 z \qquad f_r = 2rz$

check $\dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial r} + \dfrac{\partial f}{\partial z}\cdot\dfrac{\partial z}{\partial r}$

$= 2xz\cdot\cos\theta + 2yz\cdot\sin\theta + (x^2+y^2)(0)$.

$= 2rz\cos^2\theta + 2rz\sin^2\theta = 2rz$ ✓

(c) Convert the equation to spherical coordinates and compute f_ρ. Verify your answer using the chain rule.

$f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) = \rho\cos\phi(\rho^2\sin^2\phi)$

$= \rho^3\cos\phi\sin^2\phi \to f_\rho = 3\rho^2\cos\phi\sin^2\phi$

$\dfrac{\partial f}{\partial \rho} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial \rho} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial \rho} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial \rho}$

$= 2xz\cdot\sin\phi\cos\theta + 2yz\sin\phi\sin\theta + (x^2+y^2)\cos\phi$

$= 2\rho^2\sin^2\phi\cos\phi\cos^2\theta + 2\rho^2\sin^2\phi\cos\phi\sin^2\theta + \rho^2\sin^2\phi\cos\phi$

$= 2\rho^2\cos\phi\sin^2\phi + \rho^2\sin^2\phi\cos\phi = 3\rho^2\sin^2\phi\cos\phi$ ✓

6. Consider the function $f(x, y) = x^2 + y^2 + kxy$ for a constant k.

(a) For which values k does $f(x, y)$ have a local minimum at $(0, 0)$?

$f_x = 2x + ky = 0 \Rightarrow x = y = 0$ ✓ $\quad f_{xx} = 2$

$f_y = kx + 2y = 0 \quad f_{yy} = 2$

$f_{xy} = k$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$D = 4 - k^2$

local min $\Rightarrow$ $4 - k^2 > 0 \Rightarrow -2 < k < 2$

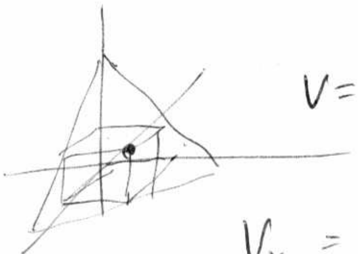
(b) For which values k does $f(x, y)$ has a saddle point at $(0, 0)$?

saddle $\Rightarrow$ $4 - k^2 < 0 \Rightarrow$ $k > 2$ or $k < -2$

(c) For which values k do we cry at $(0, 0)$?

$k = 2$ or $k = -2$

7. Using Lagrange Multipliers, find the volume of the largest rectangular box that can fit in the first octant and under the plane $ax + by + cz = d$, where a, b, c and d are positive constants.



$V = xyz \qquad g: ax+by+cz=d$

$V_x = yz = \lambda a = \lambda g_x$

$V_y = xz = \lambda b = \lambda g_y$

$V_z = xy = \lambda c = \lambda g_z$

$ax+by+cz=d$

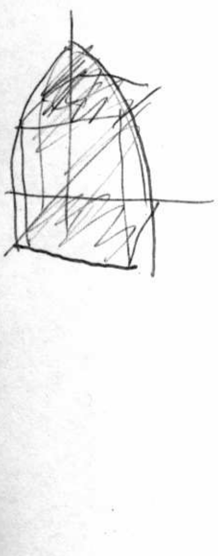
$xyz = \lambda ax = \lambda by = \lambda cz \qquad ax + b\frac{ax}{b} + c\frac{ax}{c} = d$

$y = \frac{ax}{b} \qquad 3ax = d$

$z = \frac{ax}{c} \qquad x = \frac{d}{3a} \quad y = \frac{d}{3b} \quad z = \frac{d}{3c}$

$V = \frac{d^3}{27abc}$

8. Find the volume bound by the coordinate planes, the planes $x = 1$, $y = 2$ and the paraboloid $z = 9 - x^2 - y^2$.



$$\int_0^1 \int_0^2 9 - x^2 - y^2 \, dy\, dx$$

$$= \int_0^1 \left. 9y - x^2 y - \frac{y^3}{3} \right]_0^2$$

$$= \int_0^1 18 - 2x^2 - \frac{8}{3} - 0 \, dx$$

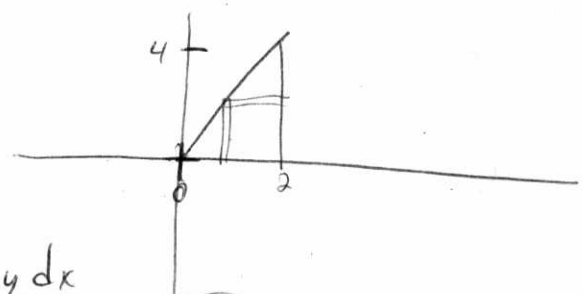
$$= \int_0^1 \frac{46}{3} - 2x^2 \, dx = \left. \frac{46}{3}x - \frac{2}{3}x^3 \right]_0^1$$

$$= \frac{46}{3} - \frac{2}{3} = \frac{44}{3}$$

9. Reverse the order of integration on

$$\int_0^4 \int_{\frac{y}{2}}^2 \cos(x^2)\, dx\, dy$$

and find its value.



$$\int_0^2 \int_0^{2x} \cos(x^2)\, dy\, dx$$

yAy!

for our 2x.

$$\int_0^2 2x\cos(x^2)\, dx$$

$$\sin(x^2)\Big]_0^2 = \sin 4 - \sin 0$$

$$= \sin(4)$$

10. (Extra Credit) Who is your Congressperson, and how did they fare in yesterday's election? If you have no Congressperson, then tell who won any race of your choice.

Cathy McMorris (R) won reelection.