KEY

Math 225: Quiz the Last
December 6, 2006

You know the drill by now. No books, no notes, no colleagues, and no answers without justification.

1. (a) What is a conservative vector field?

$\langle P, Q\rangle$ is conservative if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

also

if $\langle P, Q\rangle$, or in general, $\vec{F} = \vec{\nabla} f$ for some function $f(x,y)$

(b) What are the differences between $\int_C f(x,y)\,ds$, $\int_C f(x,y)\,dx$ and $\int_C f(x,y)\,dy$?

Integrals w.r.t. arc length, x, y

$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$ ← intuitive geometrically, hard to compute

$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\, x'(t)\,dt$ ← not intuitive, but easier.

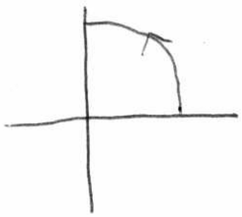
$\int_C f(x,y)\,dy =$ " $y'(t)\,dt$ ←

(c) What does it mean for a line integral to be independent of path?

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

does not depend at all on path, C.
We only care what the function is at the endpoints.

2. Compute $\int_C xy \, ds$ where C is the upper-right quarter of the unit circle centered at the origin, oriented counterclockwise.



$x = \cos t$
$y = \sin t$
$0 \le t \le \pi/2$

$$\int_0^{\pi/2} \cos t \cdot \sin t \cdot \overbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}^{1} \, dt$$

$$= \int_0^{\pi/2} \cos t \sin t \, dt$$

$$= \frac{\sin^2 t}{2} \Big]_0^{\pi/2} = \frac{1}{2}.$$

3. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 2y\mathbf{i} + 2x\mathbf{j}$ and where C is the cubic segment along $y = x^3 + 3x$ from (0,0) to (1,4)

easy:

$\vec{F} = \langle 2y, 2x \rangle$

$\frac{\partial P}{\partial y} = 2 = \frac{\partial Q}{\partial x}$

so $\vec{F}$ is conservative

$\vec{F} = \vec{\nabla} f$ where $f(x,y) = 2xy$

So $\int_C \vec{F} \cdot d\vec{r} = (2 \cdot 1 \cdot 4) - (2 \cdot 0 \cdot 0)$
$= 8.$

less easy:

$$\int_C 2y \, dx + 2x \, dy$$

$x = t$
$y = t^3 + 3t \quad 0 \le t \le 1$

$\cancel{\int_0^1 (2t^3 + 6t) 1 + 2t^3 + 6t}$

$$\int_0^1 (2t^3 + 6t) 1 + (2t)(3t^2 + 3) \, dt$$

$$= \int_0^1 8t^3 + 12t \, dt$$

$$= 2t^4 + 6t^2 \Big]_0^1 = 8.$$

4. Which of the following vector fields **F**. are conservative? For those that are, find a function $f(x, y)$ such that $\mathbf{F} = \nabla \mathbf{f}$.

   (a) $\mathbf{F} = \langle e^y, e^x \rangle$
   (b) $\mathbf{F} = \langle e^x, e^y \rangle$
   (c) $\mathbf{F} = \langle x^2 + xy, y^2 + \frac{x^2}{2} \rangle$
   (d) $\mathbf{F} = \langle x^2 + xy, y^2 + xy \rangle$

a) $\langle e^y, e^x \rangle$

$\frac{\partial P}{\partial y} = e^y$ No!

$\frac{\partial Q}{\partial x} = e^x$

b) $\langle e^x, e^y \rangle$

$\frac{\partial P}{\partial y} = 0$ ✓ yes.

$\frac{\partial Q}{\partial x} = 0$ ✓

$f_x = e^x \xrightarrow{\int dx} f = e^x + g(y)$

$f_y = e^y$

$\downarrow \frac{\partial}{\partial y}$

$f_y = 0 + g'(y)$

$g'(y) = e^y$

$g(y) = e^y + C$

$f(x,y) = e^x + e^y + C$.

c) $\langle x^2 + xy, y^2 + \frac{x^2}{2} \rangle$

$\frac{\partial P}{\partial y} = x$ $\frac{\partial Q}{\partial x} = \frac{2x}{2} = x$ ✓

$f_x = x^2 + xy \longrightarrow \int x^2 + xy \, dx = \frac{x^3}{3} + \frac{x^2}{2} y + g(y)$

$\downarrow \frac{\partial}{\partial y}$

$f_y = y^2 + \frac{x^2}{2} \longleftrightarrow \frac{x^2}{2} + g'(y)$

$y^2 = g'(y)$

$g(y) = \frac{y^3}{3} + C$

$f = \frac{x^3}{3} + \frac{x^2 y}{2} + \frac{y^3}{3} + C$

d) $\langle x^2 + xy, y^2 + xy \rangle$

$\frac{\partial P}{\partial y} = x$ $\frac{\partial Q}{\partial x} = y$ No!

5. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle x\cos(x^3) + y, x + \arctan(e^y)\rangle$, and C is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$, oriented counterclockwise.

C is closed. Let's hope $\vec{F}$ is conservative...

$$\langle x\cos(x^3)+y\ ,\ x+\arctan(e^y)\rangle$$

$$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 1 \quad \text{yay!}$$

$$\int_C \vec{F}\cdot d\vec{r} = 0$$

Extra Credit:

(a) When is your birthday? (Date only. No years. Please.) November 19

(b) Of the 40 of us, the approximate probability that two of us have the same birthday is : ($\frac{1}{90}$, $\frac{1}{4}$, $\frac{3}{4}$, or $\frac{87}{90}$).