KEY

Math 225: Quiz the Second

9·13·06

This exam is closed book and closed notes. You should not need a calculator on this quiz. You have until 5 minutes before the hour to finish.

1. Short answer

(a) The equation $x^2 + y^2 + z^2 = 4$ defines a surface of how many dimensions?

2 dimensions
(+ 1 equation → 3 total in $\mathbb{R}^3$)

(b) How do we turn a non-unit vector into a unit vector? Give an example. Also, can we always do this?

Divide each component by the length

Eg: $\langle 1,2,3 \rangle \longrightarrow \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle$ unit vector

Cannot do this if the given vector is $\vec{0}$.

(c) Complete the sentences:
If the angle between two vectors is acute, then their dot product is.... positive
If the angle between two vectors is right, then their dot product is.... 0
If the angle between two vectors is obtuse, then their dot product is.... negative
Explain.

$\vec{a} \cdot \vec{b} = \underbrace{\|\vec{a}\| \cdot \|\vec{b}\|}_{\text{positive}} \cdot \underbrace{\cos\theta}_{\text{depends on angle}}$

acute → $\cos\theta > 0$
right → $\cos\theta = 0$
obtuse → $\cos\theta < 0$

2. Consider the following equations/inequalities.

   (a) $x^2 + y^2 + (z-1)^2 = 25$

   (b) $x + y = 3$

   (c) $2x + 3y - 5z = 0$

   (d) $z \geq 0$

   (e) $x^2 + y^2 = 25$

   (a) Which equations represent planes? Which one goes through the origin?

   Equations (b) & (c) are planes.

   (c) goes through the origin.

   (b) Which is a cylinder? Which is a sphere? Explain the differences.

   (e) is a cylinder $\leftarrow$ e does not depend on z.

   (a) is a sphere

   (c) Which gives a three dimensional object?

   (d) $\rightarrow$ the "top" half of $\mathbb{R}^3$ is 3 dimensional.

3. Puxatawney Pete (Phil's autumnal cousin) comes out of his hole and his body makes the vector $\mathbf{b} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$. His shadow makes the vector $\mathbf{a} = -\mathbf{i} + 2\mathbf{j}$.

(a) Determine $\mathbf{proj_a b}$.

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{(\vec{a}\cdot\vec{b})}{\|\vec{a}\|^2}\right)\vec{a}$$

$$= \frac{\langle 1,1,-3\rangle \cdot \langle -1,2,0\rangle}{1^2+2^2+0^2}\,\vec{a}$$

$$= \frac{1}{5}\vec{a} = \left\langle \frac{-1}{5}, \frac{2}{5}, 0\right\rangle$$

$$\text{or } \frac{-1}{5}\vec{i} + \frac{2}{5}\vec{j}$$

Pete

(b) [redacted] will see his shadow only if the projection of the body vector onto his shadow vector is shorter than the shadow vector. Is it?

Projection is shorter:

$$\|\text{proj}_{\vec{a}}\vec{b}\| = \sqrt{\frac{1}{25}+\frac{4}{25}} = \frac{\sqrt{5}}{5}$$

$$\|\vec{a}\| = \sqrt{5}$$

(So 6 more weeks of summer? ☺)

4. How can we view the length of a vector in terms of a dot product?

$$\|\vec{a}\|^2 = \vec{a} \circ \vec{a}$$

so $\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$

5. Suppose **a** and **b** are orthogonal vectors. Prove that

$$\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2.$$

What is this result better known as? (Hint: Use Number 4)

$$\begin{aligned} \|\vec{a}+\vec{b}\|^2 &= (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b}) \\ &= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} \\ &= \|\vec{a}\|^2 + 2\underbrace{\vec{a}\cdot\vec{b}}_{0} + \|\vec{b}\|^2 \end{aligned}$$

But $\vec{a}\cdot\vec{b} = 0$ since $\vec{a} \perp \vec{b}$, so

$$= \|\vec{a}\|^2 + \|\vec{b}\|^2.$$

a.k.a. Pythagorean Theorem.