KEY

Math 225: Quiz the Fourth

October 4, 2006

You know the drill by now. No notes, no colleagues, and no answers without justification.

1. Let $\mathbf{r}(t)$ be a vector valued function. Identify the following as a *scalar*, a *vector*, a *scalar valued function*, or a *vector valued function*

   (a)

   $$\mathbf{r}'(t)$$

   vector valued function

   (b)

   $$\int_a^b \mathbf{r}(t)\, dt$$

   vector (not dep. on t)

   $(\vec{R}(b) - \vec{R}(a))$

   (c)

   $$||\mathbf{r}(t)||$$

   Scalar valued function

   (d)

   $$\int_0^t ||\mathbf{r}'(u)||\, du$$

   Scalar valued function (arc length)

2. Find the length of the curve given by

$$\mathbf{r}(t) = \left\langle \frac{\sqrt{8}}{3} t^{\frac{3}{2}}, t, \frac{t^2}{2} \right\rangle$$

between $1 \leq t \leq 2$.

$$\vec{r}\,'(t) = \left\langle \frac{3}{2} \cdot \frac{\sqrt{8}}{3} t^{1/2}, 1, t \right\rangle$$

$$= \langle \sqrt{2}\, t^{1/2}, 1, t \rangle$$

$$\text{Arc length} = \int_1^2 \sqrt{2t + 1 + t^2}\, dt$$

$$= \int_1^2 \sqrt{(t+1)^2}\, dt = \int_1^2 t + 1\, dt$$

$$= \left. \frac{t^2}{2} + t \right]_1^2$$

$$= 2 + 2 - \left[\tfrac{1}{2} + 1\right] = \frac{5}{2}$$

3. Give the formula for the binormal vector to a curve, $\mathbf{B}(t)$. How do we know this is a unit vector? (Hint: Think of the magnitude of a cross product)

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

unit tangent   unit normal

$T \perp N$

$$\|\vec{B}(t)\| = \|\vec{T}(t) \times \vec{N}(t)\| = \|T(t)\| \cdot \|N(t)\| \cdot \sin\theta$$

$$1 \quad 1 \quad \cdot \sin \pi/2 = 1.$$

4. Consider the curves $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 3t \rangle$.

(a) Describe the two curves geometrically, citing any similarities and differences.

Both are spirals with the z-axis as the center.

The second spins twice as fast as the first & rises 3 times as fast as the first.

(b) Show that both $\vec{r}_1(t)$ and $\vec{r}_2(t)$ have constant curvature.

$$K = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$\vec{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$

$\vec{r}_1'(t) = \langle -\sin t, \cos t, 1 \rangle \to \|r'\| = \sqrt{2}$

$r_1''(t) = \langle -\cos t, -\sin t, 0 \rangle$

$r' \times r'' = \langle -\sin t, -\cos t, 1 \rangle$ $(\sin^2 t + \cos^2 t)$

$\|r' \times r''\| = \sqrt{2}$

$K = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}$

$\vec{r}_2(t) = \langle \cos 2t, \sin 2t, 3t \rangle$

$\vec{r}_2'(t) = \langle -2\sin 2t, 2\cos 2t, 3 \rangle$

$\vec{r}_2''(t) = \langle -4\cos 2t, -4\sin 2t, 0 \rangle$

$\vec{r}' \times \vec{r}'' = \langle -12\sin 2t, -12\cos 2t, 8 \rangle$

$\|r' \times r''\| = \sqrt{144 + 64} = \sqrt{208}$

$\|r'\| = \sqrt{4+9} = \sqrt{13}$

$K = \sqrt{208} / (\sqrt{13})^3 \approx .30$

(c) Which has larger curvature and why?

The first has larger K (curvature), it is a tighter spiral. The second is more "elongated" and straighter.

E.C. Osculum → Kiss tangere → to touch