KEY

Math 225: Quiz the Eighth
November 15, 2006

You know the drill by now. No books, no notes, no colleagues, and no answers without justification.
PLEASE READ ALL OF THE QUESTIONS CAREFULLY

1. Fill in the blank

   (a) $\iint f(x,y)\;dy\;dx = \iint \underline{f(r\cos\theta, r\sin\theta)\; r}\;dr\;d\theta$

   (b) The center of mass of a plate with density function $\rho(x,y)$ is given by the formulas

$$\overline{x} = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}, \quad \overline{y} = \frac{\iint y\,\rho(x,y)\,dA}{\iint \rho(x,y)\,dA}$$

mass →

mass →

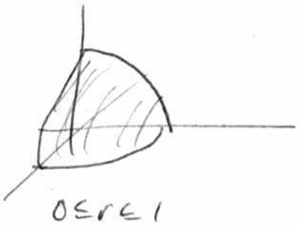
   (c) If $f(x,y)$ is a joint probability density function, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dy\;dx = 1$$

   (d) The surface area over a region D of a function $z = f(x,y)$ is given by

$$\iint_D \underline{\sqrt{1+(f_x)^2+(f_y)^2}}\;dA$$

2. Find the volume of the unit sphere that lies in the first octant.



$0 \le r \le 1$

$0 \le \theta \le \frac{\pi}{2}$

$z = \sqrt{1-x^2-y^2}$

$$\int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1-r^2}\ r\,dr\,d\theta$$

$u = 1-r^2$

$du = -2r\,dr$

$$\int_0^{\pi/2} \int_{r=0}^1 -\frac{1}{2}u^{\frac{1}{2}}\,du\,d\theta = \int_0^{\pi/2} -\frac{1}{3}(1-r^2)^{3/2}\Big]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}\,d\theta = \pi/6$$

(which is $\frac{1}{8}\left(\frac{4}{3}\pi(1)^3\right)$)

3. Find the surface area of the unit sphere that lies in the first octant (Do this in two parts. Set up the integral in x and y, then convert it to polar coordinates and solve).

$z = \sqrt{1-x^2-y^2}$

$f_x = \frac{-x}{\sqrt{1-x^2-y^2}} \qquad f_y = \frac{-y}{\sqrt{1-x^2-y^2}}$

$$\iint_R \sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}\ dA$$

$$= \iint_R \sqrt{\frac{1}{1-x^2-y^2}}\ dA = \int_0^{\pi/2}\int_0^1 \frac{1}{\sqrt{1-r^2}}\ r\,dr\,d\theta$$

$u = 1-r^2$

$du = -2r\,dr$

$$\int_0^{\pi/2} \int_{r=0}^1 -\frac{1}{2}u^{-1/2}\,du\,d\theta$$

$$= \int_0^{\pi/2} -u^{1/2}\Big]_{r=0}^1 d\theta = \int_0^{\pi/2} -\sqrt{1-r^2}\Big|_0^1 d\theta$$

$$= \int_0^{\pi/2} 1\,d\theta = \frac{\pi}{2}$$

$\left(= \frac{1}{8}(4\pi)\right)$

4. Consider a metal plate in the shape of a square, with vertices $(0, 0)$, $(4, 0)$, $(4, 4)$, and $(0, 4)$.

(a) Set up the integrals to determine the x-coordinate of the center of mass if the density is given by $\rho(x, y) = x$

$$\bar{x} = \frac{\int_0^4 \int_0^4 x^2 \, dy\,dx}{\cancel{\int_0^4 \int_0^4 xy}\,! \quad \left(\int_0^4 \int_0^4 x \, dy\,dx\right.}.$$

(b) Set up the integrals to determine the x-coordinate of the center of mass if the density is given by $\rho(x, y) = y$

$$\bar{x} = \frac{\int_0^4 \int_0^4 xy \, dy\,dx}{\int_0^4 \int_0^4 y \, dy\,dx}$$

(c) Would the values for parts (a) and (b) lie to the left, to the right, or on the line $x = 2$? Explain.

(a) to the right $\quad$ (b) on

↓ heavier for greater values of x $\quad$ ↑ density not dependent on x, should balance halfway

(I get to do both!)

(d) Verify your claim in part (c) by carrying out the integration for EITHER part (a) OR part (b)

(a) $\int_0^4 \int_0^4 x^2 \, dy\,dx$

$\int_0^4 4x^2 \, dy\,dx$

$= \frac{4x^3}{3}\Big]_0^4 = \frac{256}{3}$

b) $\int_0^4 \int_0^4 xy \, dy\,dx$

$\int_0^4 x \frac{y^2}{2}\Big]_0^4 dx$

$\int_0^4 8x \, dx$

$4x^2\Big]_0^4 = 64$

$\int_0^4 \int_0^4 x \, dy\,dx \longleftrightarrow \int_0^4 \int_0^4 y \, dy\,dx$

same

$\int_0^4 4x \, dx$

$= 2x^2\Big]_0^4 = 32$

$\bar{x}_a = \frac{256/3}{32}$

$= 8/3 > 2.$

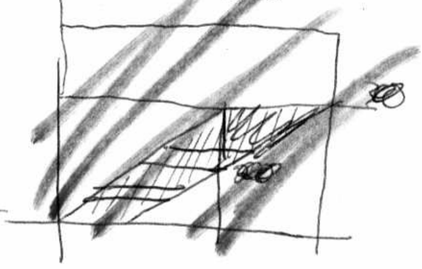
32

$\bar{x}_b = 64/32 = \underline{2}$

5. Suppose that two buses, the Xylophone Line and the Youngstown Line, are due to arrive at a stop. They will arrive sometime between noon and 1PM, with all times being equally likely for each bus. The bus will arrive at the stop, wait 10 minutes, and then depart. What is the probability that the ~~Xeriscape~~ Xylophone Line arrives before 12:30 and that the Youngstown Line has to wait for it.

(Hint: The probability distribution function is given by $f(x,y) = \frac{1}{3600}$ for $0 \le x \le 60$ and $0 \le y \le 60$, and 0 everywhere else. Also: You'll get most of the credit for setting up the bounds.)

$P(0 \le X \le 30 \,;\, \text{[illegible]})$



$x \le Y \le x+10$

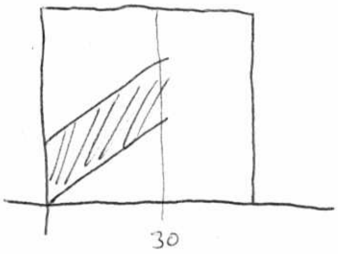
~~Need to break up the Integral~~

$$\int_0^{30} \int_x^{x+10} \frac{1}{3600} \, dY \, dX$$

$$= \int_0^{30} \frac{1}{3600} \quad x+10-x \; dx$$

$$= \int_0^{30} \frac{1}{3600} \cdot 10 \, dx$$

$$= \int_0^{30} \frac{1}{360} dx = \frac{30}{360} = 1/12$$



Extra Credit: You may take half of a point for yourself, or zero points for yourself and give $\frac{1}{20}$ of a point to everyone else (38 others besides yourself). Your extra credit will be the total number of points chosen by you for yourself and by others for you.

The Key gives $\frac{1}{20}$ of a point around.