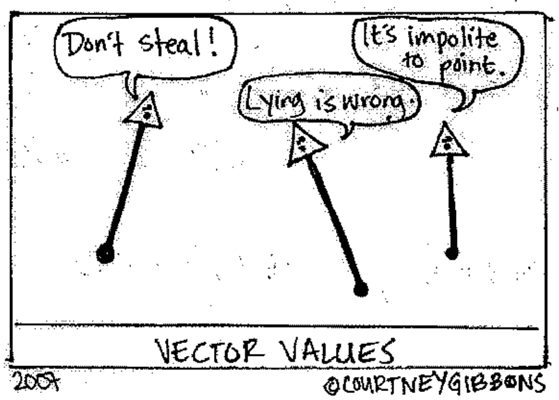
Key

Math 225: Exam the First

You have two hours to complete this exam. You may use a calculator for computation only, and you should be prepared to show the relevant steps to a problem where necessary.



1. (a) Give the equation of the sphere with radius 4 centered at the point $(0, 0, 3)$.

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$$(x-0)^2+(y-0)^2+(z-3)^2=16$$

$$x^2+y^2+(z-3)^2=16$$

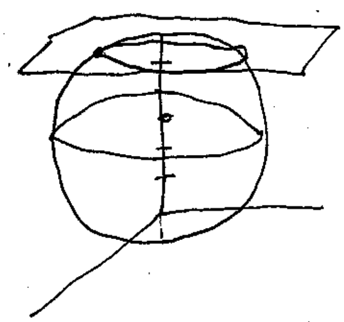
(b) What is the intersection of this sphere with the plane $z = 5$? Give your answer as both a description and as a parametric equation.

If $z=5$

Then

$x^2+y^2+(5-3)^2=16$

$x^2+y^2=12$



Circle centered @ $(0,0,5)$ with radius $= 2\sqrt{3}$

$x = 2\sqrt{3}\cos t$

$y = 2\sqrt{3}\sin t$

$z = 5$

2. (a) Find the equation of the plane that contains the points $\overset{A}{(4,1,1)}, \overset{B}{(5,3,3)}$ and $\overset{C}{(7,1,2)}$.

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3 points → 2 vectors

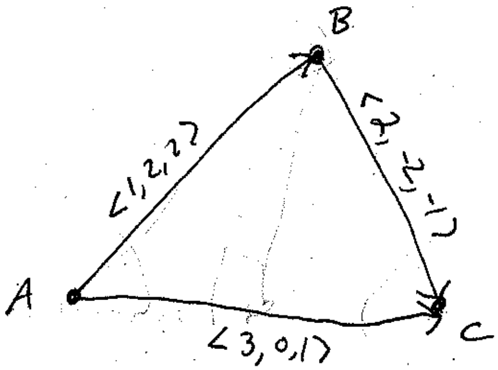
$\vec{AB} = \langle 5-4, 3-1, 3-1 \rangle = \langle 1,2,2 \rangle$

$\vec{AC} = \langle 7-4, 1-1, 2-1 \rangle = \langle 3,0,1 \rangle$

so the plane has $\vec{n} = \vec{AB} \times \vec{AC} = \begin{array}{r} \langle 1,2,2 \rangle \\ \times \langle 3,0,1 \rangle \\ \hline \langle 2,5,-6 \rangle \end{array}$

plane: $2(x-4)+5(y-1)-6(z-1)=0$

(b) Is the triangle determined by these three points acute, right, or obtuse? Explain.



$\measuredangle A$ measured by $\vec{AB} \cdot \vec{AC} = \langle 1,2,2 \rangle \langle 3,0,1 \rangle$
$= 5 > 0$, so
$\measuredangle A$ Acute

$\measuredangle B$ measured by $\vec{BA} \cdot \vec{BC} = \langle -1,-2,-2 \rangle \cdot \langle 2,-2,-1 \rangle$
$= -2+4+2 = 4 > 0$
so $\measuredangle B$ Acute.

$\measuredangle C$ measured by $\vec{CA} \cdot \vec{CB} = \langle -3,0,-1 \rangle \cdot \langle -2,2,1 \rangle$
$= 6+0-1 = 5 > 0$
so $\measuredangle C$ acute

so $\triangle ABC$ is Acute

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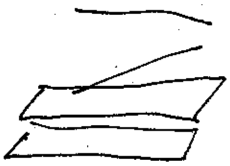
3. True or False, and <u>why</u>?

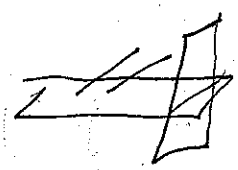
(a) Two planes parallel to each of two skew lines are themselves parallel.

True. We can compute the normal to each plane by crossing the direction vectors of the skew lines, which are different.



(b) Two planes parallel to each of two parallel lines are themselves parallel.

False. Here, there is no cross product to take (the cross of the direction vectors is $\vec{0}$), so we can have planes which are parallel to both, yet intersect



(c) Two planes can intersect in exactly one point.

False. Two planes are parallel, or intersect in a line.

(d) Three planes can intersect in exactly one point

True. For example xy, yz, xz-planes all intersect at the origin.

(e) Four planes can intersect in exactly one point.

True. Take for example xy, xz, yz planes and the plane $x+y+z=0$.

Again, all 3 go through the origin

(15)

4. Given here are a list of equations. Which represent the same surface when converted between coordinate systems?

   (a) $x^2 + y^2 - 4z^2 = 1$
   (b) $x^2 + y^2 = 16$
   (c) $y = \sqrt{3}x$
   (d) $\rho^2(\sin^2(\phi) - 4\cos^2\phi) = 1$
   (e) $r^2 + z^2 = 8z$
   (f) $\rho = 8\cos(\phi)$
   (g) $\rho\sin(\phi) = 4$
   (h) $\theta = \frac{\pi}{3}$
   (i) $x^2 + y^2 + (z-4)^2 = 16$
   (j) $r = 4$

$a = d$

$x^2+y^2-4z^2=1$

$r^2 - 4z^2 = 1$

$\rho^2\sin^2\phi - 4\rho^2\cos^2\phi = 1$

$b = g = j$

$x^2+y^2=16$

$r^2 = 16$

$r=4$

$\rho\sin\phi = 4$

$i = e = f$

$x^2+y^2+(z-4)^2=16$

$r^2+z^2-8z+16=16$

$r^2+z^2=8z$

$\rho^2 = 8\rho\cos\phi$

$\rho = 8\cos\phi$

$c = h$

$y = \sqrt{3}\,x$

$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{\sqrt{3}x}{x}\right)$

$\theta = \arctan(\sqrt{3})$

$=$

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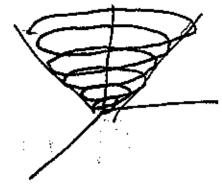
5. The function $\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$ has a 'conical helix' as its graph.



(a) Find the equation of the tangent line to this helix when $t = \frac{\pi}{2}$

$$\vec{r}(t) = \langle t\cos t, t\sin t, t\rangle$$

$$\vec{r}(\pi/2) = \langle \pi/2 \cos\tfrac{\pi}{2}, \tfrac{\pi}{2}\sin\tfrac{\pi}{2}, \pi/2\rangle$$

$$= \langle 0, \tfrac{\pi}{2}, \tfrac{\pi}{2}\rangle$$

$$\vec{r}\,'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 1\rangle$$

$$\vec{r}\,'(\tfrac{\pi}{2}) = \langle \cos\tfrac{\pi}{2} - \tfrac{\pi}{2}\sin\tfrac{\pi}{2}, \sin\tfrac{\pi}{2} + \tfrac{\pi}{2}\cos\tfrac{\pi}{2}, 1\rangle$$

$$= \langle -\tfrac{\pi}{2}, 1, 1\rangle$$

line: $\vec{\ell}(t) = \langle 0, \frac{\pi}{2}, \frac{\pi}{2}\rangle + t\langle -\frac{\pi}{2}, 1, 1\rangle$ or $x = -\frac{\pi}{2}t$, $y = \frac{\pi}{2} + t$, $z = \frac{\pi}{2} + t$.

(b) Set up, but don't compute, an integral which calculates the arc length of this curve on its first rotation up from the origin.

1st rotation: $t = 0$ to 2π

$$\int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}\;dt$$

Simplification optional (but useful)

$$\int_0^{2\pi} \sqrt{\cos^2 t \cancel{- 2t\sin t\cos t} + t^2\sin^2 t + \sin^2 t \cancel{+ 2t\sin t\cos t} + t^2\cos^2 t + 1}\;dt$$

$$= \int_0^{2\pi} \sqrt{1 + t^2 + 1}\;dt = \int_0^{2\pi} \sqrt{2 + t^2}\;dt$$

(c) Describe what happens to the curvature as t increases. DO NOT try to calculate an explicit formula for the curvature.

As t increase, the size of the circle gets larger & larger, hence the vector $\vec{T}(t)$ is changing slower & slower

so $K \to 0$ as $t \to \infty$

15 6. (a) Give the formula for the binormal vector to a curve, $\mathbf{B}(t)$.

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

(b) Differentiate your formula to find a formula for $\mathbf{B}'(t)$.

$$\vec{B}'(t) = \left(\vec{T}(t) \times \vec{N}(t)\right)'$$
$$= \vec{T}'(t) \times \vec{N}(t) + \vec{T}(t) \times \vec{N}'(t)$$

$=0$, since $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$

so $\vec{B}'(t) = \vec{T}(t) \times \vec{N}'(t)$

(c) Prove that $\mathbf{B}'(t)$ is perpendicular to $\mathbf{T}(t)$.

Since $\vec{B}'(t)$ is $\vec{T}(t)$ crossed with $\vec{N}'(t)$, $\vec{B}'(t)$ is a vector perpendicular to both $\vec{T}(t)$ & $\vec{N}'(t)$.

Hence $\vec{T}(t) \perp \vec{B}'(t)$.

(Also $\vec{T} \cdot (\vec{T} \times \vec{N}') = 0$)

(10) 7. Prove that a path that has zero acceleration is either a straight line or a single point.

Suppose $\vec{a}(t) = \langle 0, 0, 0 \rangle$

then $\vec{v}(t) = \int_0^t \vec{a}(u)\,du + \langle c_1, c_2, c_3 \rangle$ ← init velocity

$= \langle c_1, c_2, c_3 \rangle$

← init position

so $\vec{s}(t) = \langle \int_0^t c_1\,du, \int_0^t c_2\,du, \int_0^t c_3\,du \rangle + \langle d_1, d_2, d_3 \rangle$

$= \langle c_1 t + d_1, c_2 t + d_2, c_3 t + d_3 \rangle$

which is the equation of the line through $\langle d_1, d_2, d_3 \rangle$ in the direction of $\langle c_1, c_2, c_3 \rangle$ ✓, ″, etc.

Note that if all $c_i = 0$, then the path stays at the point $\langle d_1, d_2, d_3 \rangle$.

(+3) 8. (Bonus) Give the mathematical terms that come from:

(a) the Latin for 'to kiss' osculating plane

(b) the Latin for 'to touch' tangent (line, plane, etc.)

(c) the Greek for 'to throw alongside' Parabola

alongside to throw