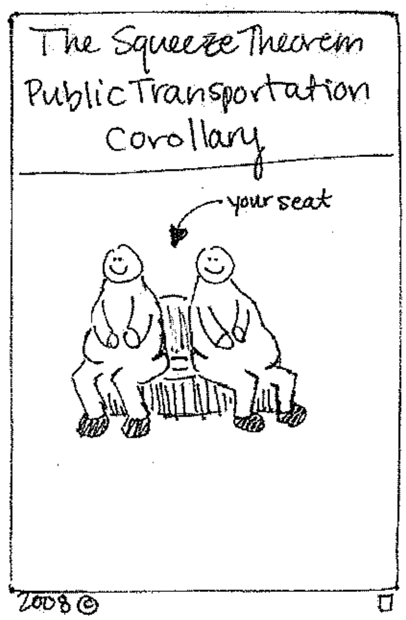
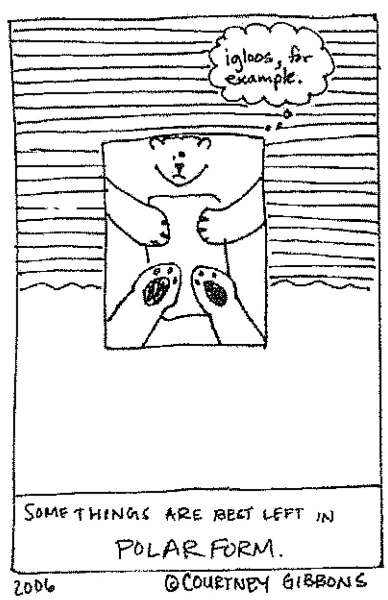
KEY

Math 225: Exam the Second

You have two hours to complete this exam. You may use a calculator for basic arithmetic and trig/exponential functions only, and you should be prepared to show the relevant steps to a problem where necessary.





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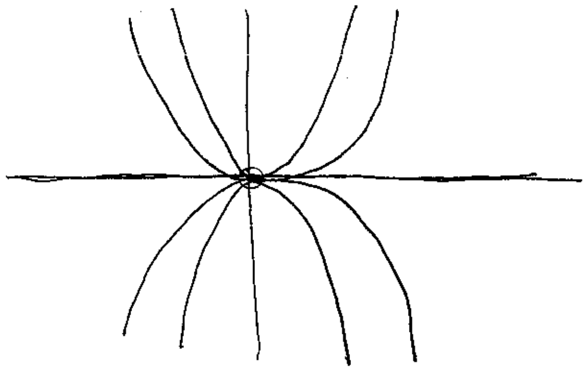
1. (a) Draw the level curves to $f(x,y) = \frac{y}{x^2}$ for $f(x,y) = -2, -1, 0, 1, 2$.

$k = \frac{y}{x^2} \rightarrow y = kx^2$

"parabolas"



(b) What is happening with these level curves at the origin?

Though it appears that all parabolas go through the origin, we know this cannot be the case.

$f(0,0)$ is undefined, so each parabola misses the origin.

10

2. Find, if it exists, the following limit

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x^2+y^2)-1}{(x^2+y^2)^2}$$

Convert to polars

$$\lim_{r\to 0} \frac{\cos(r^2)-1}{(r^2)^2} = \lim_{r\to 0} \frac{\cos(r^2)-1}{r^4}$$

$$\xrightarrow{L'H} \lim_{r\to 0} \frac{-2r\sin(r^2)}{4r^3}$$

$$= \lim_{r\to 0} \frac{-\sin(r^2)}{2r^2} = -\frac{1}{2}$$

10

3. Find the mixed partial derivative f_{xy} of the function $f(x,y) = \ln(x)\cos(x) + y^3e^{2y^2} + xy$. (Hint: Try to save yourself some work here.)

$f_x(x,y) =$ (some messy function in x) $+ 0 + y$

$f_{xy}(xy) = 0 + 0 + \boxed{1}$

The first two parts of the derivative disappear when we take <u>both</u> partials.

15

4. Consider the function $f(x, y) = x^4y - x^2y^3$

(a) Find the direction in which f *decreases* the fastest from the point $(2, -3)$.

$\vec{\nabla} f = \langle 4x^3y - 2xy^3,\ x^4 - 3x^2y^2 \rangle$

$= \langle 32(-3) - 4(-27),\ 16 - 3\cdot4\cdot9 \rangle$

$= \langle 12, -92 \rangle$ Dir of max decrease $\rightarrow \langle -12, 92 \rangle$

(b) Find the tangent plane and use it to approximate $f(2.02, -2.97)$.

$f(2,-3) = 16(-3) - 4(-27)$

$= 60$

Tangent plane

$f_x(2,-3) = 12$

$f_y(2,-3) = \cancel{92}\ -92$

$z = 60 + 12(x-2) - 92(y+3)$

Approx:

$f(2.02, -2.97) \approx 60 + 12(2.02-2) - 92(-2.97+3)$

$\approx 60 + .24 - 2.76 = 57.48$

(c) Consider the level curve $f(x, y) = 60$.

i. Find $\frac{dy}{dx}$ for this curve.

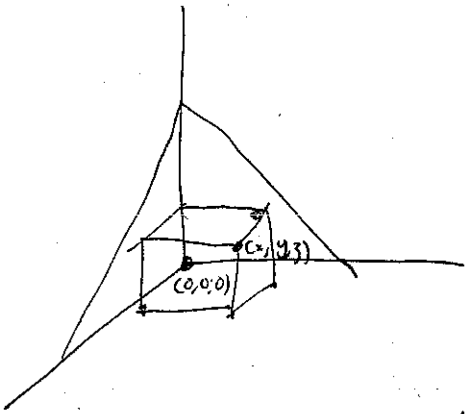
$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(4x^3y - 2xy^3)}{(x^4 - 3x^2y^2)}$$

ii. Evaluate $\frac{dy}{dx}$ at $(2, -3)$. How does this value relate to your answer from part (a)?

$$\left.\frac{dy}{dx}\right|_{(2,-3)} = \frac{-12}{-92} = \frac{12}{92}$$

which is the negative reciprocal of the "slope" of the gradient.

5. Find the maximum volume of a box in the first octant with opposite corners at the origin and on the plane $x + 2y + 3z = 10$, respectively. (You may use any method that you wish).



$V = xyz$ subject to $x + 2y + 3z = 10$

$\vec{\nabla} V = \lambda \vec{\nabla}(\text{plane})$

$$\left.\begin{aligned} yz &= \lambda \\ xz &= 2\lambda \\ xy &= 3\lambda \end{aligned}\right\} \text{mult. by the missing}$$

$\lambda x = 2y\lambda = 3z\lambda$

$x = 2y = 3z \rightarrow x + x + x = 10$

$3x = 10$

$x = \frac{10}{3}$

$y = \frac{10}{6}$

$z = \frac{10}{9}$

max volume $= \frac{1000}{3 \cdot 6 \cdot 9} = \underline{\frac{1000}{162}}$

10

6. Find the volume bound by the surface $z = ye^{xy}$ and the planes $x = 0, x = 1, y = 0, y = 1$ and $z = 0$.

easier order

$$\int_0^1 \int_0^1 y e^{xy} \, dx \, dy$$

$$\int_0^1 \left[e^{xy} \right]_0^1 dy$$

$$= \int_0^1 e^y - 1 \, dy = e^y - y \Big|_0^1 = (e-1) - 1$$

$$= e - 2$$

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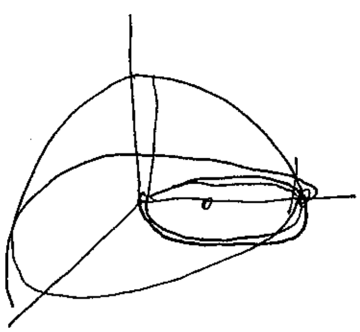
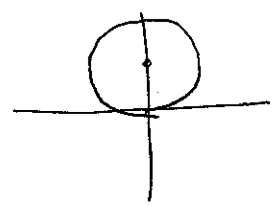
7. Find the volume above the xy plane bound by $x^2 + y^2 + z^2 = 4$ and $x^2 + (y-1)^2 = 1$



polars: $\mathbb{R} \rightarrow$  $0 \leq 2\sin\theta \leq r$

$0 \leq \theta \leq \pi$

$$\int_0^\pi \int_0^{2\sin\theta} \sqrt{4-r^2} \; r \, dr \, d\theta \qquad u = 4 - r^2 \quad du = -2r\,dr$$

$$= \int_0^\pi -\frac{1}{3}(4-r^2)^{3/2} \Big|_0^{2\sin\theta} d\theta$$

$$= \int_0^\pi -\frac{1}{3}\left((4 - 4\sin^2\theta)^{3/2} - 8\right) d\theta$$

$$= \int_0^\pi -\frac{1}{3}(4\cos^3\theta - 8)\, d\theta$$

$$= -\frac{\cancel{8}}{3}\int_0^\pi (\cancel{1} - \cancel{\sin}^2\theta)\cos\theta - \cancel{8}\, d\theta$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

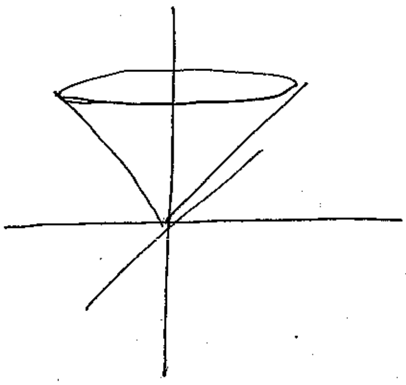
$$\rightarrow -\frac{1}{3}\left[\sin\theta - \frac{\sin^3\theta}{3} - 8\theta\right]_0^\pi$$

$$= \cancel{\frac{1}{3}\cos\theta}$$

$$= \frac{8}{3}\pi.$$

5

8. Find the volume above the cone $z = r$ and below the plane $z = h$.



$$\int_0^{2\pi}\int_0^{h} (h-r)\, r\,dr\,d\theta$$

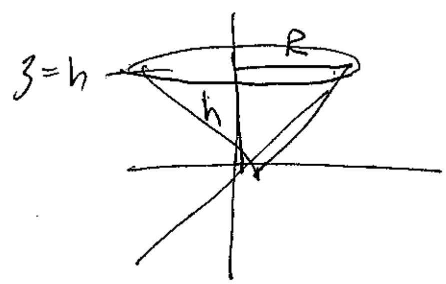
$$= \int_0^{2\pi} \left[\frac{hr^2}{2} - \frac{r^3}{3}\right]_0^h d\theta$$

$$= \int_0^{2\pi} \frac{1}{6}h^3\, d\theta = \underline{\frac{\pi}{3}h^3}$$

9. The volume bound by the cone $z = \frac{h}{R}r$ and the plane $z = h$ is a cone of radius R and height h. Find, using calculus, the volume of this cone.

Intersection: $\frac{h}{R}r = h \qquad \frac{r}{R} = 1 \qquad r = R$

$$\int_0^{2\pi}\int_0^{R} \left(h - \frac{h}{R}r\right) r\,dr\,d\theta$$



$$= \int_0^{2\pi}\int_0^{R} hr - \frac{h}{R}r^2\, dr\,d\theta$$

$$= \int_0^{2\pi} \frac{hr^2}{2} - \frac{h}{R}\frac{r^3}{3}\Big|_0^R d\theta$$

$$= \int_0^{2\pi} \frac{hR^2}{2} - \frac{h}{R}\frac{R^3}{3}\, d\theta$$

$$= \int_0^{2\pi} \frac{hR^2}{6}\, d\theta = \pi\frac{hR^2}{3} = \boxed{\frac{1}{3}\pi R^2 h}$$

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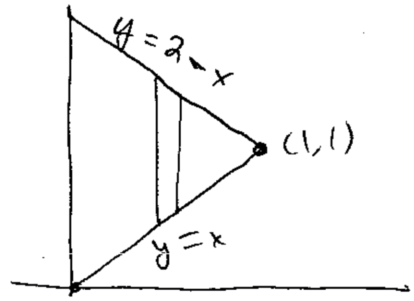
10. Consider a plate in the shape of a triangle with vertices (0,0), (1,1) and (0,2) and density $\rho(x, y) = x$.

(a) Find the mass of a plate.



$$\int_0^1 \int_x^{2-x} x \, dy\, dx$$

$$= \int_0^1 xy \Big|_x^{2-x} dx$$

$$= \int_0^1 (2x - x^2) - x^2 \, dx = \int_0^1 2x - 2x^2 \, dx = \left[\frac{2x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= x^2 - \frac{2x^3}{3} \Big|_0^1 = \frac{1}{3}$$

(b) Set up the integral to find $\overline{x}$

$$\overline{x} = \frac{\iint_R x\rho(x,y)\, dx}{\iint \rho(x,y)\, dx} = \frac{\int_0^1 \int_x^{2-x} x^2 \, dy\, dx}{\frac{1}{3}} = 3\int_0^1 \int_x^{2-x} x^2 \, dy\, dx$$

(c) Find $\overline{y}$ without calculation.

$\overline{y} = 1$, as the plate & density function balance on $y = 1$.

11. (Extra Credit) Fill in the grey box in the following cartoon...