Math 225: Exam the Second

You have two hours to complete this exam. You may use a calculator for computation only, and you should be prepared to show the relevant steps to a problem where necessary.

1. Consider the functions $f(x,y) = x^2 + y^2$, $g(x,y) = \sqrt{x^2+y^2}$, and $h(x,y) = \ln(x^2+y^2)$.
   (a) What familiar surfaces are the graphs of f and g?
   (b) What familiar curve is a level curve for f, g, or h?
   (c) Plot the level curves for each of the three functions for $z = 1, 2, 3, 4$ and 5, with attention to spacing to show steepness.
   (d) Which function is steepest? Which is least steep? Explain.

2. Find
$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$
if it exists.

3. Suppose that for a continuous and differentiable function $f(x,y)$, we know that $f_x(x,y) = 2xy + x^3$. Give at least three possible functions that could be $f_y(x,y)$.

4. Suppose that the temperature of a plate is given by the function $f(x,y) = \frac{y}{x^2}$.
   (a) Suppose you are at the point $(2,3)$. In what direction should you move so as to increase your temperature the most rapidly?
   (b) Suppose that you move from $(2,3)$ towards $(5,-1)$. At what rate is your temperature changing?
   (c) Approximate the temperature at $(2.04, 2.99)$.

5. Find and classify the critical points of $f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} - \frac{x^2}{2} - y^2$. (There are four of them.)

6. Find the minimum surface area of a box without a top that has volume 27.

7. Find the following integrals. Where necessary or appropriate, you may reverse the order of integration or convert into another coordinate system.
   (a)
$$\int_0^1 \int_0^1 y\cos(xy)dy\,dx$$
   (b)
$$\int_1^e \int_{\ln x}^1 e^{e^y} dy\,dx$$
   (For (b), reverse, but don't calculate)
   (c)
$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^2 + y^2 dy\,dx$$

8. Consider a plate in the shape of a quarter circle of radius 1 in the first quadrant. Suppose that the density of the plate is given by $\rho(x, y) = (x - y)^2$.

   (a) Find the mass of the plate.

   (b) Set up the integral to find $\overline{x}$. Convert this integral to polar coordinates, but DO NOT attempt to compute it.

   (c) Argue that, for this plate, $\overline{y} = \overline{x}$.

9. Find the volume of the region bounded by the planes $y = 0, x = 1, z = 0, y = x^2$ and $4x + 2y + z = 7$

10. Find, using Calculus, the volume of a sphere of radius R.