KEY

Math 225: Quiz the Eighth

You know the drill by now. No book, no notes, no colleagues, and no answers without justification.

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1. Short Answer

   (a) Fill in the blanks

   $$dz\,dy\,dx = \underline{\quad r \quad}\,dz\,dr\,d\theta = \underline{\quad \rho^2 \sin\phi \quad}\,d\rho\,d\theta\,d\phi$$

   (b) Suppose that for a u, v-substitution, the Jacobian equals a constant, k. What is the geometric relationship between the two regions of integration (the x, y-region and the u, v-region)? Note: Word your answer carefully.

   The (u, v) region is $\frac{1}{k}$'th the size of the (x, y) region

   (c) Give parametric equations for the line segment from (a, b) to (c, d). Be sure to state relevant t values.

   $x = a + (c-a)t$

   $y = b + (d-b)t$

   $0 \le t \le 1$

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2. Calculate

$$\iiint_E x\,dV$$

where V is bound above by $z=2$, below by $z=x^2+y^2$, and on the sides by $y=0$, $y=x$,and $x=1$.

$$\int_0^1\int_0^x\int_{x^2+y^2}^{2} x\;dz\,dy\,dx$$

$$=\int_0^1\int_0^x 2x-x^3-xy^2\,dy\,dx$$

$$=\int_0^1 2xy-x^3y-x\frac{y^3}{3}\Big]_0^x dx=\int_0^1 2x^2-\frac{4}{3}x^4\,dx$$

$$=\frac{2}{3}x^3-\frac{4}{15}x^5\Big|_0^1=\frac{2}{3}-\frac{4}{15}=\frac{6}{15}=\frac{2}{5}.$$

3. Calculate

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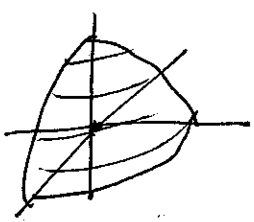
$$\iiint_E e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV$$

where E is the portion of the unit sphere centered at the origin contained in the first octant.



$0\le\rho\le 1$

$0\le\theta\le \pi/2$

$0\le\phi\le\frac{\pi}{2}$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 e^{\rho^3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$u=\rho^3$

$du=3\rho^2\,d\rho$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{3}e^{\rho^3}\Big]_0^1\sin\phi\,d\phi\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{3}(e-1)\sin\phi\,d\phi\,d\theta$$

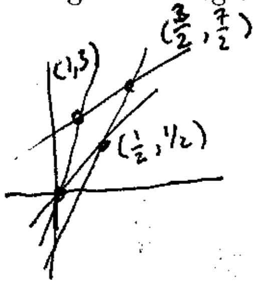
$$=\int_0^{\pi/2}\frac{1}{3}(e-1)(-\cos\phi)\Big|_0^{\pi/2}d\theta=\int_0^{\pi/2}\frac{1}{3}(e-1)\,d\theta$$

$$=\frac{\pi}{6}(e-1)$$

5 4. Consider the following integral:

$$\iint_R x + y \, dA, \text{ where } R \text{ is bound by } y = x, y = 3x, y = x + 2, y = 3x - 1$$

(a) Sketch the region of integration. What shape is it? What are the corners?



(b) Consider the substitution:

$$x = \frac{u+v}{2}; \; y = \frac{3u+v}{2}.$$

Solve this substitution for u and v and explain why it is an appropriate substitution.

$y - x = u$

$3x - y = v$

$0 \le u \le 2$

$0 \le y \le 1$

$\mathcal{J} : \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

$= \left|\frac{1}{4} - \frac{3}{4}\right| = \frac{1}{2}$

(c) Make the u, v substitution, take all necessary steps, and find the integral.

$x + y = \frac{u+v}{2} + \frac{3u+v}{2} = 2u + v$

$\int_0^2 \int_0^1 2u + v\left(\frac{1}{2}\right) dv\, du = \frac{1}{2}\int_0^2 \left[2uv + \frac{v^2}{2}\right]_0^1 du$

$= \frac{1}{2}\int_0^2 2u + \frac{1}{2}\, du = \frac{1}{2}\left[u^2 + \frac{1}{2}u\right]_0^2 = \frac{4+1}{2} = 5/2$

4 5. Calculate

$$\int_C x^2\,dx + xy\,dy$$

where C is the parabolic segment given by $y = x^2$ from $(2,4)$ to $(4,16)$.

$x = t$
$y = t^2$
$2 \le t \le 4$

$$\int_2^4 t^2 + 2t^4\,dt = \left.\frac{t^3}{3} + \frac{2t^5}{5}\right]_2^4$$

$$= \frac{64}{3} + \frac{2048}{5} - \frac{8}{3} - \frac{64}{5}$$

$$= \frac{56}{3} + \frac{1984}{5}$$

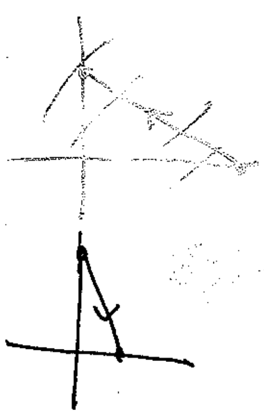
3 6. Calculate

$$\int_C 2x + y\,ds,$$

where C is the line segment from $(0,2)$ to $(1,0)$, and explain why the integral simplifies so dramatically.

$$\int_0^1 (2t + 2 - 2t)\,\sqrt{5}\,dt \qquad \begin{aligned} x &= 0 + t \\ y &= 2 - 2t \end{aligned}$$

$$= 2\sqrt{5}.$$



It simplifies, since on this line segment

$y = -2x + 2$

or $\underset{\uparrow}{y + 2x} = 2$

integrand.

7. Extra Credit: You may take one, two, three or four points. Those giving the *least* popular answer will receive their chosen amount of extra credit.