KEY

Math 225: Exam the First

You have 90 minutes to complete this exam. READ ALL DIRECTIONS CAREFULLY. You may use a calculator for computation only, and you should be prepared to show the relevant steps to a problem where necessary.

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1. Let $A = (2, 3, 1)$, $B = (6, 1, 5)$, and let $P = (x, y, z)$

   (a) Find the distance from A to B.

   $D(A,B) = \sqrt{(6-2)^2 + (1-3)^2 + (5-1)^2} = \sqrt{16+4+16} = \sqrt{36} = 6$

   (b) Write the following statement as an equation:

   The distance from P to A is the same as the distance from P to B.

   $$\sqrt{(x-2)^2 + (y-3)^2 + (z-1)^2} = \sqrt{(x-6)^2 + (y-1)^2 + (z-5)^2}$$

   (c) Simplify your equation and describe the set of points (x, y, z) that satisfy the description in part (b).

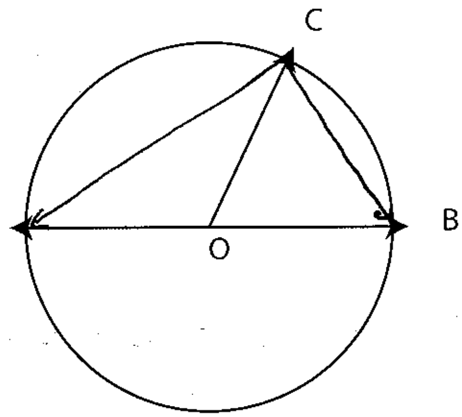
   $$(x-2)^2 + (y-3)^2 + (z-1)^2 = (x-6)^2 + (y-1)^2 + (z-5)^2$$

   $$\cancel{x^2} - 4x + 4 + \cancel{y^2} - 6y + 9 + \cancel{z^2} - 2z + 1 = \cancel{x^2} - 12x + 36 + \cancel{y^2} - 2y + 1 + \cancel{z^2} - 10z + 25$$

   $$8x - 4y + 8z = 48$$

   it is a plane!

2. Consider the following diagram:



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In this picture, A and B are on opposite sides of a diameter, C is any other point on the circle, and O is the center. Prove, using vectors, that angle ACB is a right angle.

Want to show $\vec{CA} \cdot \vec{CB} = 0$

$\vec{CA} = \vec{CO} + \vec{OA}$ $\qquad$ $\vec{CB} = \vec{CO} + \vec{OB}$

$= \vec{CO} - \vec{OA}$

$(\vec{CO} + \vec{OA}) \cdot (\vec{CO} - \vec{OA}) =$

$\vec{CO} \cdot \vec{CO} + \vec{CO} \cdot \vec{OA} - \vec{CO} \cdot \vec{OA} - \vec{OA} \cdot \vec{OA}$

$= |\vec{CO}|^2 - |\vec{OA}|^2$, but, since both are radii, their magnitudes are equal

$= 0$

3. Consider the following two line equations:

$$\ell_1(t) = \langle 1+3t, t, 2-2t \rangle; \; \ell_2(t) = \langle -6t+3, -2t+3, 4t \rangle$$

(a) Explain why ℓ_1 is parallel to ℓ_2.

dir $\vec{v}$ for l_1 $\langle 3,1,-2 \rangle$

dir $\vec{v}$ for l_2 $\langle -6,-2,4 \rangle = -2\langle 3,1,-2 \rangle$, so they run in the same direction and are parallel.

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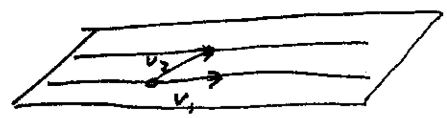
(b) Find the plane that contains ℓ_1 and ℓ_2.

point $\langle 1,0,2 \rangle$

$\vec{n}$ $\quad \vec{v}_1 \quad \langle 3,1,-2 \rangle$

$\times \vec{v}_2 \quad \langle 2,3,-2 \rangle$

$\langle 4,2,7 \rangle$



v_2: $(1,0,2)$

$(3,3,0)$

$\langle 2,3,-2 \rangle$

plane: $4(x-1)+2y+7(z-2)=0$.

4. Suppose that we have three lines, each of which is skew to the other two. Might there be a plane parallel to all 3 lines? Must there be a plane parallel to all 3 lines? Explain.

There might be, but there need not be

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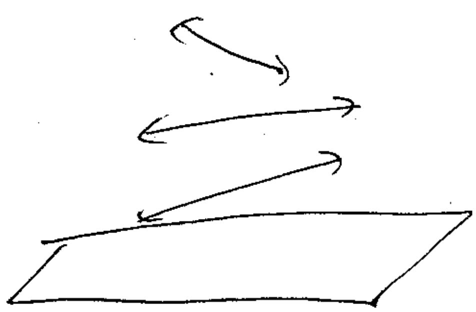
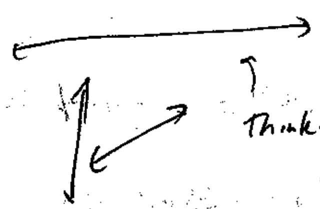
Ex



three skew lines all "flat" with regard to the plane



Think of this as set from the back.

now, no plane is parallel to all 3?

5. Consider the equation $\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos(2t) \rangle$.

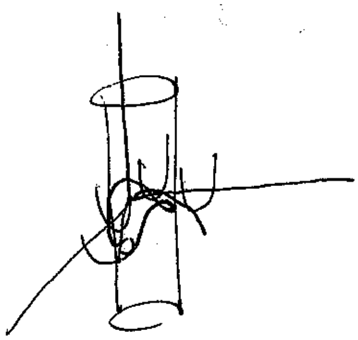
(a) This curve is the intersection of which two surfaces? (Here, you'll need algebraic relations between x, y and z components. Try a trig identity for the relation with z, which I can 'sell' to you if necessary.) Be sure that you give the names of the surfaces as well as the algebraic equations.

$x^2+y^2=1 \rightarrow$ cylinda

$\cos(2t) = \cos^2 t - \sin^2 t$

$z = x^2 - y^2 \rightarrow$ hyperbolic paraboloid

(b) Give a rough sketch of the curve. (You may give a prose description to aid you here.)



(c) Find the equation of the tangent line to $\mathbf{r}(t)$ when $t = \frac{\pi}{2}$.

$\vec{r}\left(\frac{\pi}{2}\right) = \cancel{\langle 0, 1, 1 \rangle}\ \langle 0, 1, -1 \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 2\sin 2t \rangle$

$\vec{r}\,'\left(\frac{\pi}{2}\right) = \langle -1, 0, 0 \rangle$

line $\vec{l}(t) = \langle -1, 0, 0 \rangle t + \langle 0, 1, -1 \rangle$

6. Suppose that the position of an object is given by $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.

(a) Find the velocity and acceleration of the particle as a function of time.

$$\vec{v}(t) = \langle 2t, e^t, e^t + te^t \rangle$$

$$\vec{a}(t) = \langle 2, e^t, e^t + e^t + te^t \rangle$$
$$= \langle 2, e^t, 2e^t + te^t \rangle$$

(b) Find the speed of the particle as a function of time. Is the speed ever 0?

speed $= |\vec{v}(t)| = \sqrt{4t^2 + e^{2t} + (t+1)^2 e^{2t}}$

no, since $e^{2t} > 0$ for all t,

Speed is never 0.

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(c) Set up, but don't compute, the integral for the distance that the particle travels from $t = 1$ to $t = 2$.

$$\int_1^2 \sqrt{4t^2 + e^{2t}[1 + (t+1)^2]}\, dt$$

7. Match the equation to the surface description. Warning: There is one outlier in each group!!

(a) $x^2 + 9y^2 + 81z^2 = 81$ III
(b) $x^2 + 9y^2 + 81 = 81z^2$ II
(c) $x^2 + 9y^2 + 81 = 81z$ IV
(d) $x^2 - 9y^2 + 81 = 81$
none.

I Hyperboloid of One Sheet
II Hyperboloid of Two Sheets
III Ellipsoid
IV Elliptical Paraboloid

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8. For vectors $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$, we have that

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$$

Use this formula and the definition of the binormal vector, $\mathbf{B}(t)$, to prove that $\mathbf{N}(t) \times \mathbf{B}(t) = \mathbf{T}(t)$.

$$N(t) \times B(t) = N(t) \times (T(t) \times N(t))$$

$$= \underbrace{(N \cdot N)}_{|N|^2 = 1} T - \underbrace{(N \cdot T)}_{0 \text{, since } N \perp T} N$$

$$= \vec{T}.$$

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9. (Bonus) Prove that if $\mathbf{r}(t)$ has zero curvature for all t, then $\mathbf{r}(t)$ is either a straight line or a single point.

$$\kappa(t) = 0 \Rightarrow T'(t) = \vec{0}$$

$$\Rightarrow \vec{T}(t) = \langle c_1, c_2, c_3 \rangle \leftarrow \text{unit}$$

$$r'(t) = \langle c_1', c_2', c_3' \rangle \leftarrow \text{might not be unit}$$

$$\Rightarrow \vec{r}(t) = \langle c_1' t + d_1, c_2' t + d_2, c_3' t + d_3 \rangle$$

which is the line through $\langle d_1, d_2, d_3 \rangle$ in the direction of $\langle c_1', c_2', c_3' \rangle$