KEY

Math 225: Exam the Second

You have 90 to complete this exam. You may use a calculator for basic arithmetic and trig/exponential functions only, and you should be prepared to show the relevant steps to a problem where necessary.

1. Let $f(x, y) = \frac{1}{1+x^2+y^2}$

   (a) For which values x, y is f defined?

   Since $1+x^2+y^2 \geq 1 > 0$, the denominator is never 0.

   Thus, f is defined for all x, y

15

   (b) For which values $f(x, y) = k$ can we find level curves?

   $K = \frac{1}{1+x^2+y^2}$

   We know k is never negative, since the denominator is never negative, thus $K > 0$

   Solving for a familiar form

   $\frac{1}{K} = 1+x^2+y^2 \quad x^2+y^2 = \frac{1}{K} - 1 \geq 0$

   thus $\frac{1}{K} \geq 1$ or $K \leq 1$

   So we have $1 \geq K > 0$

   (c) What are the shape of these level curves?

   These level curves are of the form

   $x^2+y^2 = \frac{1}{K} - 1$, thus are circles.

2. Find

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$$

if it exists.

$\frac{y^2}{x^2+y^2}$ is a proper fraction (denom $\geq$ num)

thus we have $0 \leq \frac{y^2}{x^2+y^2} \leq 1$

So $0 \leq \frac{2xy^2}{x^2+y^2} \leq 2x$

and since $0, 2x \to 0$ as $x, y \to 0$

we have squeezed

$\frac{2xy^2}{x^2+y^2}$ to 0

10

3. Suppose that you are hiking in a valley with equation $f(x,y) = (x-2)^2 + (2y-3)^2$. You spill from your water bottle at the point (1,0). In what direction will the water begin to travel (note: water travels directly downhill)?

Looking for direction of max decrease from $(1,0)$ of F.

need $-\vec{\nabla} f$

$\vec{\nabla} f = \langle 2(x-2), 2(2y-3)(2) \rangle \Big|_{(1,0)}$

$= \langle -2, -12 \rangle$

So Downhill $\to$ $\langle 2, 12 \rangle$

10

4. The total resistance of two resistors with resistances x and y in parallel is given by

$$R = \frac{xy}{x+y}.$$

Approximate the total change in resistance if x goes from 2 to 2.5 and y goes from 3 to 2.7.

We need the tangent plane @ $(2,3)$ $\quad R(2,3) = \frac{6}{5}$

$$R_x = \frac{(x+y)(y) - xy(1)}{(x+y)^2} \qquad R_x(2,3) = \frac{15-6}{25} = \frac{9}{25}$$

10

$$R_y = \frac{(x+y)(x) - xy}{(x+y)^2} \qquad R_y(2,3) = \frac{10-6}{25} = \frac{4}{25}$$

$$R \approx \frac{6}{5} + \frac{9}{25}(x-2) + \frac{4}{25}(y-3)$$

$$R(2.5, 2.7) \approx \frac{6}{5} + \frac{9}{25}\cdot\frac{1}{2} + \frac{4}{25}\cdot\left(\frac{-3}{10}\right)$$

$$\frac{60}{50} + \frac{9}{50} - \frac{12}{250} = \frac{300}{250} + \frac{45}{250} - \frac{12}{250} = 333/250$$

Change $= 33/250$

5. Find and classify the critical points of $f(x,y) = xy^2 - 3x^3 + 6y$. (There are two of them).

$$f_x = y^2 - 9x^2 = 0 \qquad y^2 = 9x^2 \qquad y = \pm 3x$$

$$f_y = 2xy + 6 = 0$$

$y = 3x \Rightarrow 6x^2 + 6 = 0$ $\leftarrow$ no solution

15

$y = -3x \Rightarrow -6x^2 + 6 = 0$ $\leftarrow x = \pm 1$ $\quad y = \mp 3$

cp's $\quad (1, -3) \quad (-1, 3)$

saddle points

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

$$= (-18x)(2x) - (2y)$$

opposite signs

6. A function $z = f(x, y)$ satisfying

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

is called a *harmonic* function. Prove that $f(x, y) = x^3 - 3xy^2$ is harmonic.

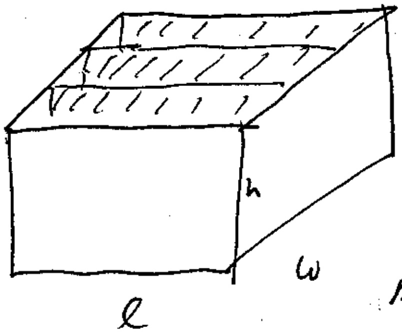
$f_x = 3x^2 - 3y^2$ $\quad$ $f_y = -6xy$

$f_{xx} = 6x$ $\quad$ $f_{yy} = -6x$

so $f_{xx} + f_{yy} = 6x - 6x = 0$.

5

7. A shipping carton without a top is to have two dividers, as shown in the figure on the board. If we have 48 square meters of material to use, what are the dimensions of the box that will maximize volume. (You may ignore the thickness of the material in computing volume).



bottom + sides (left, right) + (front, back, dividers)

$S = lw + 2wh + 4lh = 48$

maximize $V = lwh$ subj. to $lw + 2wh + 4lh = 48$

$\nabla V = \lambda \nabla S$

$V_h = lw = (2w + 4l)\lambda \Leftrightarrow 2wh + 4lh$

$V_w = lh = (l + 2h)\lambda \Leftrightarrow lw + 2wh$

$V_l = wh = (w + 4h)\lambda \Leftrightarrow lw + 4lh$

multiply by missing parts

15

$lw = 4lh \Rightarrow w = 4h$

$2wh = 4lh \Rightarrow w = 2l$

$w = 2l = 4h$

$l = 2h$

$S(h) = (2h)(4h) + 2(4h)(h) + 4(2h)(h)$

$= 24h^2 = 48$

$h^2 = 2$, ~~$l = 4, w = 8$~~ $h = \sqrt{2}$, $l = 2\sqrt{2}$, $w = 4\sqrt{2}$

8. Find the volume bound by $z = xy + 3x + 2y$, and the planes $z = 0$, $x = 0$, $x = 2$, $y = 1$, $y = 3$.

$$\int_0^2 \int_1^3 xy + 3x + 2y \, dy \, dx$$

$$= \int_0^2 \frac{xy^2}{2} + 3xy + y^2 \Big|_1^3 \, dx$$

$$= \int_0^2 \left(\frac{9}{2}x + 9x + 9\right) - \left(\frac{1}{2}x + 3x + 1\right) dx$$

$$= \int_0^2 10x + 8 \, dx = 5x^2 + 8x \Big|_0^2 = 20 + 16 = 36$$

8

9. Find

$$\int_{-1}^{1} \int_{2}^{4} xy^2 \cos(y) \, dy \, dx$$

(You may find Fubini's Theorem helpful.)

Fubini: $\int_2^4 \int_{-1}^1 (y^2 \cos y) x \, dx \, dy$

$$= \int_2^4 (y^2 \cos y) \frac{x^2}{2} \Big|_{-1}^1 \, dy$$

$$= \int_2^4 y^2 \cos y \left(\frac{1}{2} - \frac{1}{2}\right) dy = \int_2^4 y^2 \cos y \, (0) \, dy = 0$$

7

10. On the practice test we proved

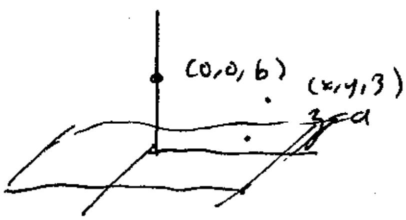
> The set of all points that are equidistant from a given point and a given line form a parabola.

This statement pertained to $\mathbb{R}^2$. Make a similar statement pertaining to $\mathbb{R}^3$.

The set of all points that are equidistant from a given point and a given PLANE form a PARABOLOID

5

11. (Bonus) Prove your statement, using $(0, 0, b)$ and $z = a$.



$$d\big((x,y,z),(0,0,b)\big) = d\big((x,y,z), \text{plane } z=a\big)$$

$$\sqrt{x^2+y^2+(z-b)^2} = z-a$$

$$x^2+y^2+(z-b)^2 = (z-a)^2$$

$$x^2+y^2+\cancel{z^2}-2zb+b^2 = \cancel{z^2}-2za+a^2$$

$$x^2+y^2+b^2-a^2 = 2zb-2za$$

or $$z = \frac{x^2+y^2+b^2-a^2}{2b-2a}$$ A PARABOLOID!