KEY

1

Math 225: Exam the Second

You have two hours to complete this exam. You may use a calculator for computation only, and you should be prepared to show the relevant steps to a problem where necessary.

1. Consider the functions $f(x,y) = x^2 + y^2$, $g(x,y) = \sqrt{x^2+y^2}$, and $h(x,y) = \ln(x^2+y^2)$.

   (a) What familiar surfaces are the graphs of f and g?

   $f \to$ a paraboloid

   2 $g \to$ a cone

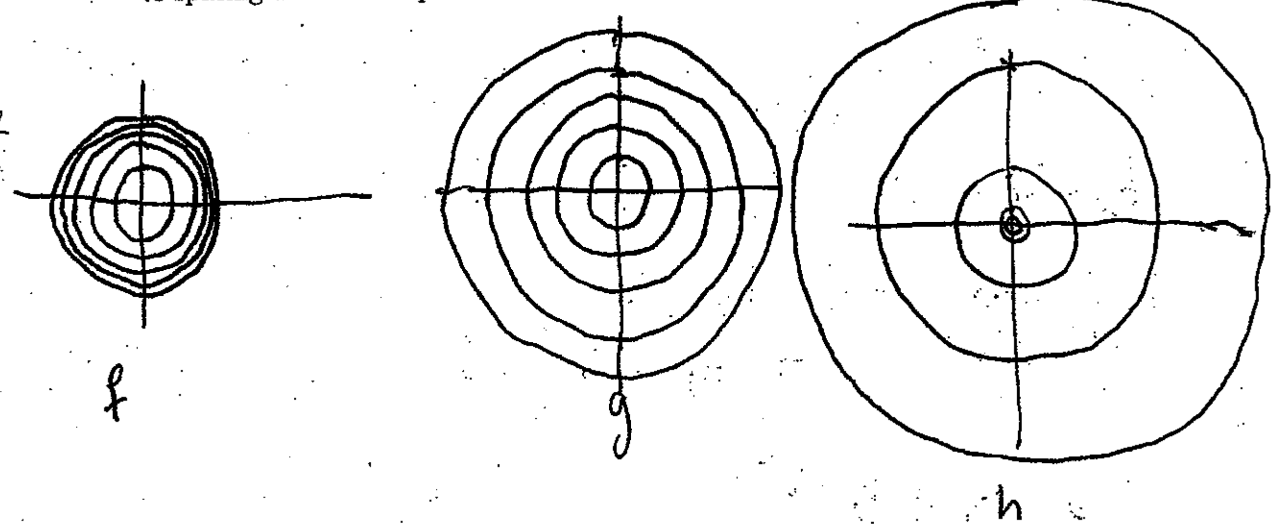
   (b) What familiar curve is a level curve for f, g, or h?

   For all 3, the curve is a circle

   2 $K = x^2+y^2$ $\quad K^2 = x^2+y^2$ $\quad e^K = x^2+y^2$

   (c) Plot the level curves for each of the three functions for $z = 1, 2, 3, 4$ and 5, with attention to spacing to show steepness.

2



   (d) Which function is steepest? Which is least steep? Explain.

   2 f is steepest
   curves closer together

   h is least steep
   curves farther apart

2

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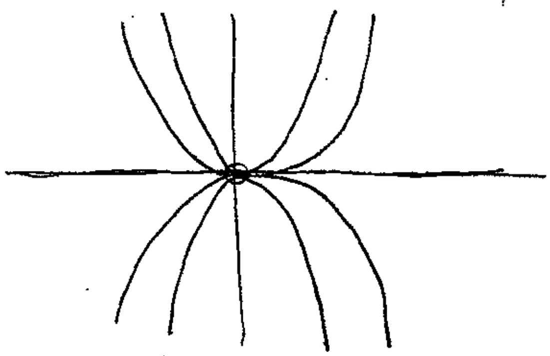
1. (a) Draw the level curves to $f(x,y) = \frac{y}{x^2}$ for $f(x,y) = -2, -1, 0, 1, 2$.

$k = \frac{y}{x^2} \rightarrow y = kx^2$

"parabolas"



(b) What is happening with these level curves at the origin?

Thought it appears that all parabolas go through the origin, we know this cannot be the case.

$f(0,0)$ is undefined, so each parabola misses the origin.

3

2. Find

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$

if it exists.

If $x=0$ $\lim_{y\to 0} = \frac{(-y)^2}{y^2} = 1$

so limit DNE.

If $x=y$ $\lim_{y\to 0} \frac{0}{2y^2} = 0$

4

5

3. Suppose that for a continuous and differentiable function $f(x,y)$, we know that $f_x(x,y) = 2xy + x^3$. Give at least three possible functions that could be $f_y(x,y)$.

$f_x = 2xy + x^3$

$f_{xy} = 2x \longrightarrow f_{yx} = 2x$

$\longrightarrow f_y = x^2 + g(y)$

6

So any of f_y x^2

$f_y = x^2 + \cos y$

$f_y = x^2 + \arcsin(e^y)$ work

4

10

2. Find, if it exists, the following limit

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x^2+y^2)-1}{(x^2+y^2)^2}$$

Convert to polars

$$\lim_{r\to 0} \frac{\cos(r^2)-1}{(r^2)^2} = \lim_{r\to 0} \frac{\cos(r^2)-1}{r^4}$$

$$\overset{L'H}{\longrightarrow} \lim_{r\to 0} \frac{-2r\sin(r^2)}{4r^3}$$

$$= \lim_{r\to 0} \frac{-\sin(r^2)}{2r^2} = -\frac{1}{2}$$

6

10

3. Find the mixed partial derivative f_{xy} of the function $f(x,y) = \ln(x)\cos(x) + y^3e^{2y^2} + xy$. (Hint: Try to save yourself some work here.)

$f_x(x,y) = (\text{some messy function in } x) + 0 + y$

$f_{xy}(x,y) = 0 + 0 + \boxed{1}$

The first two parts of the derivative disappear when we take <u>both</u> partials.

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4. Suppose that the temperature of a plate is given by the function $f(x, y) = \frac{y}{x^2}$.

(a) Suppose you are at the point $(2, 3)$. In what direction should you move so as to increase your temperature the most rapidly?

$f(x,y) = \frac{y}{x^2}$

$f_x = \frac{-2y}{x^3}$ @ $(2,3)$ $\to f_x = \frac{-3}{4}$

5 $f_y = \frac{1}{x^2}$ $\to f_y = \frac{1}{4}$

so, in the direction of $\langle \frac{-3}{4}, \frac{1}{4} \rangle$

(b) Suppose that you move from $(2, 3)$ towards $(5, -1)$. At what rate is your temperature changing?

$(2,3) \to (5,-1)$ has vector $\langle 3, -4 \rangle$

or $\vec{u} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$

$\langle \frac{-3}{4}, \frac{1}{4} \rangle \langle \frac{3}{5}, \frac{-4}{5} \rangle = \frac{-9}{20} + \frac{-4}{20} = \frac{-13}{20}$ units T / units d

5

(c) Approximate the temperature at $(2.04, 2.99)$.

$f(2,3) = \frac{3}{4}$

$z \approx z_0 + f_x (x - x_0) + f_y (y - y_0)$

6 $= \frac{3}{4} + \left(\frac{-3}{4}\right)(2.04 - 2) + \frac{1}{4}(2.99 - 3)$

$= \frac{3}{4} + \frac{-.12}{4} + \frac{-.01}{4}$

~~3.1~~ $\boxed{\frac{2.87}{4}}$

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4. Consider the function $f(x,y) = x^4y - x^2y^3$

(a) Find the direction in which f *decreases* the fastest from the point $(2, -3)$.

$$\vec{\nabla} f = \langle 4x^3y - 2xy^3,\ x^4 - 3x^2y^2 \rangle$$

$$= \langle 32(-3) - 4(-27),\ 16 - 3\cdot 4\cdot 9 \rangle$$

$$= \langle 12, -92 \rangle \qquad \text{Dir of max decrease} \to \langle -12, 92 \rangle$$

(b) Find the tangent plane and use it to approximate $f(2.02, -2.97)$.

$$f(2,-3) = 16(-3) - 4(-27)$$
$$= 60$$

Tangent plane

$$f_x(2,-3) = 12 \qquad z = 60 + 12(x-2) - 92(y+3)$$
$$f_y(2,-3) = \cancel{92}\ -92$$

Approx:
$$f(2.02, -2.97) \approx 60 + 12(2.02-2) - 92(-2.97+3)$$
$$\approx 60 + .24 - 2.76 = 57.48$$

(c) Consider the level curve $f(x,y) = 60$.

i. Find $\frac{dy}{dx}$ for this curve.

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(4x^3y - 2xy^3)}{(x^4 - 3x^2y^2)}$$

ii. Evaluate $\frac{dy}{dx}$ at $(2, -3)$. How does this value relate to your answer from part (a)?

$$\left.\frac{dy}{dx}\right|_{(2,-3)} = \frac{-12}{-92} = \frac{12}{92}$$

which is the negative reciprocal of the "slope" of the gradient.

9. $f(x,y,z) = xy + z^2 \quad \langle x,y,z \rangle = \langle t, t^2, t^4 \rangle$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \qquad t=1 \quad \langle x,y,z \rangle = 1,1,1$$

$$= (y)(1) + x(2t) + 2z(4z^3)$$

$$= 1 + 2 + 8 = 11$$

10. $\frac{dl}{dt} = 1 \quad \frac{dw}{dt} = -0.5 \quad \frac{dh}{dt} = 2$

$V = lwh \qquad l=3, w=4, h=5$

$$\frac{dV}{dt} = lw\frac{dh}{dt} + lh\frac{dw}{dt} + wh\frac{dl}{dt}$$

$$= 12(2) + 15(-0.5) + 20(1)$$

$$= 36.5 \ in^3/min$$

See correction on last page

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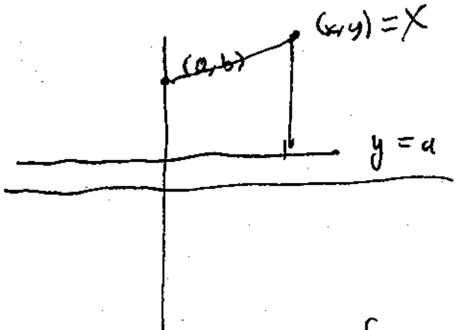

$d(X, \text{line}) = y - a$

$d(X, B) = \sqrt{x^2 + (y-b)^2}$

so if $y - a = \sqrt{x^2 + (y-b)^2}$

then $(y-a)^2 = x^2 + (y-b)^2$

$y^2 - 2ay + a^2 = x^2 + y^2 - 2yb + b^2$

$-2ay + a^2 = x^2 - 2yb + b^2$

$(2b - 2a)y = x^2 + b^2 - a^2$

$$y = \frac{x^2}{2b-2a} + \frac{b^2 - a^2}{2b-2a} \leftarrow \underline{\text{Parabola}}$$

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5. Find and classify the critical points of $f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} - \frac{x^2}{2} - y^2$. (There are four of them.)

$f_x = x^2 - x = 0 \quad x = 0, 1$

$f_y = y^2 - 2y = 0 \quad y = 0, 2$

15 $f_{xx} = 2x - 1$

$f_{yy} = 2y - 2$

$f_{xy} = 0$

points

$(0,0) \to$ MAX

$(1,0) \to$ S.P.

$(0,2) \to$ S.P.

$(1,2) \to$ min

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= (2x-1)(2y-2)$

since $f_{xx} < 0$

$D(0,0) = (-1)(-2) > 0 \Rightarrow$ max or ~~min~~

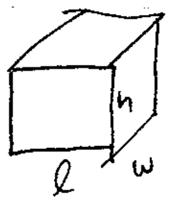
$D(1,0) = (1)(-2) < 0 \to$ Saddlepoint

$D(0,2) = (-1)(2) < 0 \to$ Saddle point

$D(1,2) = (1)(2) > 0 \to$ ~~max~~ or min

since $f_{xx} > 0$

12. $S.A. = lw + 2wh + 2lh$ $\qquad V = lwh = 27$



Lagrange

$S_w: l + 2h = lh \longleftrightarrow ① \ lw + 2wh = wlh$

$S_l \ \ w + 2h = wh \longleftrightarrow ② \ wl + 2lh = lwh$

$S_h \ \ 2l + 2w = lw \longleftrightarrow ③ \ 2lh + 2wh = hlw$

$lwh = 27$

$① = ② \quad lw + 2wh = wl + 2lh$

$\Rightarrow l = w$

$wl + 2lh = 2lh + 2wh \Rightarrow 2h = l$

$h = \frac{1}{2} l$

$l = w = 2h$

$lwh = 27 \ ; \ 4h^3 = 27$

$\Rightarrow h = \frac{3}{\sqrt[3]{4}} \ , \ w = \frac{6}{\sqrt[3]{4}} = l .$

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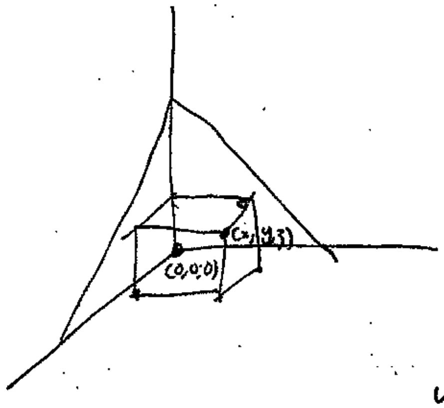
5. Find the maximum volume of a box in the first octant with opposite corners at the origin and on the plane $x + 2y + 3z = 10$, respectively. (You may use any method that you wish).



$V = xyz$ subject to $x + 2y + 3z = 10$

$\vec{\nabla} V = \lambda \vec{\nabla}(\text{plane})$

$$\left.\begin{aligned} yz &= \lambda \\ xz &= 2\lambda \\ xy &= 3\lambda \end{aligned}\right\} \text{ mult. by the missing}$$

$\lambda x = 2y\lambda = 3z\lambda$

$x = 2y = 3z \rightarrow x + x + x = 10$

$3x = 10$

$x = \frac{10}{3}$

$y = \frac{10}{6}$

$z = \frac{10}{9}$

max volume $= \frac{1000}{3 \cdot 6 \cdot 9} = \underline{\frac{1000}{162}}$

15 a) $\int_0^2 \int_0^4 yx^2 + xy + y^2 \, dy\, dx$

$= \int_0^2 \frac{y^2x^2}{2} + x\frac{y^2}{2} + \frac{y^3}{3} \Big|_0^4 dx$

$= \int_0^2 8x^2 + 8x + \frac{64}{3} \, dx$

$\frac{8}{3}x^3 + \frac{8x^2}{2} + \frac{64}{3}x \Big]_0^2$

$\frac{64}{3} + 16 + \frac{128}{3} = 64 + 16 = 80$

b) $\int_0^1 \int_0^1 y\cos(xy)\, dy\, dx$

$= \int_0^1 \int_0^1 y\cos(xy)\, dx\, dy$

$= \int_0^1 \sin x \, dy$

$-\cos x \Big|_0^1 = 1 - \cos 1$

16. $\int_2^4 \int_1^3 y e^{xy} \, dx\, dy$

$= \int_2^4 e^{xy} \Big|_1^3 dy$

$= \int_2^4 e^{3y} - e^y \, dy = \frac{e^{3y}}{3} - e^y \Big|_2^4$

$= \left(\frac{e^{12}}{3} - e^4\right) - \left(\frac{e^6}{3} - e^2\right)$

10. $\frac{dh}{dt} = 1 \quad \frac{dw}{dt} = -0.5 \quad \frac{dl}{dt} = 2$

$l = 3,\ w = 4\ \ h = 5$

$V = wlh$

$$\frac{dV}{dt} = \frac{\partial V}{\partial h}\frac{dh}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial l}\frac{dl}{dt}$$

$$= wl(1) + hl(-0.5) + wh(2)$$

$$= 12(1) - 15(-0.5) + 20(2)$$

$$= 44.5\ m^3/min$$

$S = 2wl + 2wh + 2lh$

$$\frac{dS}{dt} = \frac{\partial S}{\partial h}\frac{dh}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial l}\frac{dl}{dt}$$

$$= 2(l+w)(1) + 2(l+h)(-0.5) + 2(w+h)(2)$$

$$= 14 - 8 + 36 = 42\ m^2/min$$