FALL 2009 Review Sheet Key

1.a) $A = (1,1,2)$ $\quad \vec{x} = \vec{AC} = \langle 2,3,4 \rangle$

$B = (2,3,4)$ $\quad y = \vec{AB} = \langle 1,2,2 \rangle$

$C = (3,4,6)$

plane: point $(1,1,2)$

$\vec{n} = \vec{x} \times \vec{y} = \langle 2,3,4 \rangle \times \langle 1,2,2 \rangle = \langle -2, 0, 1 \rangle$ $\quad$ plane: $-2(x-1) + (z-2) = 0$

b) Area $= \frac{1}{2}$ (p-gram) $= \frac{1}{2} |\vec{x} \times \vec{y}| = \frac{\sqrt{5}}{2}$

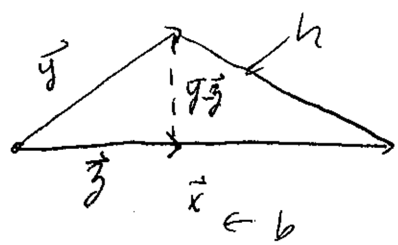
c) $\text{proj}_{\vec{x}} \vec{y} = \frac{\vec{y} \cdot \vec{x}}{|\vec{x}|^2} \vec{x}$

$$= \frac{\langle 2,3,4 \rangle \cdot \langle 1,2,2 \rangle}{|\langle 2,3,4 \rangle|^2} \langle 2,3,4 \rangle = \frac{16}{29} \langle 2,3,4 \rangle = \left\langle \frac{32}{29}, \frac{48}{29}, \frac{64}{29} \right\rangle$$

d) $\frac{1}{2} |\vec{x}| \, |\vec{y} - \vec{z}| = \frac{1}{2} \sqrt{29} \cdot \frac{\sqrt{9+100+36}}{29} = \frac{1}{2} \sqrt{29} \, \frac{\sqrt{145}}{29} = \frac{\sqrt{5}}{2}$

$(\vec{y} - \vec{z}) = \left\langle \frac{-3}{29}, \frac{10}{29}, \frac{-6}{29} \right\rangle$

Area of a triangle $= \frac{1}{2} b \cdot h = \frac{1}{2} |\vec{x}| \, |\vec{y} - \vec{z}|$



2 a) $A = (1,2,3)$ $\quad B = (-1,0,4)$ $\quad C = (2,-3,5)$

$l_{AB} = \vec{r}(t) = \langle 1,2,3 \rangle + t \langle -2,-2,1 \rangle$

$l_{AC} = \vec{r}(t) = \langle 1,2,3 \rangle + t \langle 1,-5,2 \rangle$

Angle $= \arccos\left(\frac{\langle -2,-2,1 \rangle \cdot \langle 1,-5,2 \rangle}{|\langle -2,-2,1 \rangle| \, |\langle 1,-5,2 \rangle|} \right)$

$= \arccos\left(\frac{10}{3\sqrt{30}} \right)$

b) $\vec{n} = \langle -2,-2,1 \rangle \times \langle 1,-5,2 \rangle = \langle 1, 5, 12 \rangle$ $\quad$ plane: $(x-1) + 5(y-2) + 12(z-3) = 0$

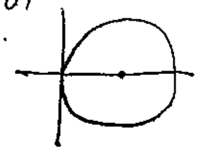
c) line normal $\quad$ point $(2,-3,5)$, vector $\langle 1,5,12 \rangle$

$x = 2 + t$

$y = -3 + 5t$

$z = 5 + 12t$

3. a)



$x = \cos t + 1$

$y = \sin t$

$0 \le t \le 2\pi$

b)

$(x-1)^2 + y^2 = 1$

$\rightarrow r = 2\cos\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

c) Circumference → Arclength

$= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$

$= 2\pi$

d) $\text{Area} = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r \, dr\, d\theta$

$= \int_{-\pi/2}^{\pi/2} 2\cos^2\theta \, d\theta$

$= \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta = \theta + \frac{\sin 2\theta}{2} \Big]_{-\pi/2}^{\pi/2}$

$= \frac{\pi}{2} + 0 - \left(-\frac{\pi}{2} + 0\right) = \pi$

4. $\langle \cos t, \sin t, \cos 2t \rangle = \vec{r}(t)$ (FROM EXAM 1)

a) $-1 \le x, y, z \le 1$

b) $\vec{r}\,'(t) = \langle -\sin t, \cos t, -2\sin 2t \rangle$

$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 4\sin^2 2t}$

$= \sqrt{1 + 4\sin^2 2t} \ne 0$

so, yes, the curve is smooth

c) $x^2 + y^2 = 1$ Cylinder

$z = x^2 - y^2$ hyperbolic paraboloid

5) $\vec{r}(t) = \langle \ln t, 2\sqrt{t}, t^2 \rangle$ $\quad \vec{r}(1) = \langle 0, 2, 1 \rangle$

$\vec{r}\,'(t) = \langle \frac{1}{t}, \frac{1}{\sqrt{t}}, 2t \rangle$ $\quad \vec{r}\,'(1) = \langle 1, 1, 2 \rangle$

Line: $x = 0 + t$
$y = 2 + t$
$z = 1 + 2t$

@ $(0, 2, 1)$ $t = 1$

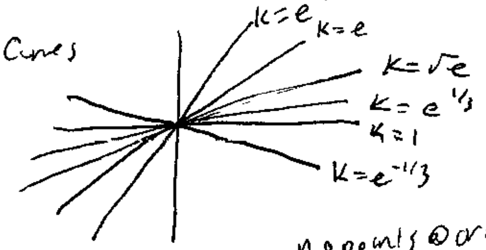
Calculate $\vec{T}(t)$; $\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$; $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

normal plane: $\vec{T}$, point

osculating plane: $\vec{B}$, point

6) $e^{y/x} = K \rightarrow \frac{y}{x} = \ln k \rightarrow y = (\ln k)\, x$

Curves



Surface is undefined @ x=0 (y-axis)

as y increases → f grows exponentially

y increase → f "asymptotes" to 0

no points @ origin, f undefined

7) a) $f(x,y) = \frac{1}{x^2+y^2+1}$ (from EXAM 2)

f defined for all x, y (no division by 0)

b) $k = \frac{1}{x^2+y^2+1} \Rightarrow K > 0$

$1 \geq K > 0$

$x^2+y^2 = \frac{1}{K} - 1 \Rightarrow K \leq 1$

← Curves are circles.

c) Tplane @ (2, −1)

$f(2,-1) = \frac{1}{6}$

$f_x(x,y) = \frac{-2x}{(x^2+y^2+1)^2}$ $f_x(2,-1) = -\frac{4}{6} = -\frac{2}{3}$

$f_y(x,y) = \frac{-2y}{(x^2+y^2+1)^2}$ $f_y(2,-1) = \frac{2}{6} = \frac{1}{3}$

plane: $z = \frac{1}{6} - \frac{2}{3}(x-2) + \frac{1}{3}(y+1)$

$f(1.97, -.97) \approx \frac{1}{6} - \frac{2}{3}(1.97-2) + \frac{1}{3}(-.97+1)$

$= \frac{1}{6} - \frac{2}{3}(-.03) + \frac{1}{3}(-.03)$

$= \frac{1}{6} + \frac{.03}{3} = \frac{1}{6} + .01$

8) $f(x,y) = e^x \cos(xy)$

To approximate $f(.01, .02)$ find Tplane @ (0,0)

$f(0,0) = 1$

$f_x(0,0) = e^x\cos(xy) + e^x(-\sin(xy)(y))\Big|_{(0,0)} = 1$

$f_y(0,0) = e^x(x\cos(xy))\Big|_{(0,0)} = 0$

plane = $z = 1 + 1(x)$

Approx ~~f(0.01)~~ $f(.01, .02) = 1 + (.01) = 1.01$

9) $f(x,y,z) = \tan(x+2y+3z)$

$\vec{\nabla} f = \langle \sec^2(x+2y+3z), 2\sec^2(x+2y+3z), 3\sec^2(x+2y+3z) \rangle$

$\vec{\nabla} f\Big|_{(5,-1,-1)} = \langle 1, 2, 3 \rangle$ ← dir. of max change

NOTE: $\vec{\nabla} f \perp \langle -5, 1, 1 \rangle$

$D_{\vec{u}} f$ as $\vec{u}$ moves to origin

$\vec{u} = \frac{\langle -5,1,1 \rangle}{|\langle -5,1,1 \rangle|} = \frac{\langle -5,1,1 \rangle}{3\sqrt{3}}$

$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$

$= \langle 1,2,3 \rangle \cdot \frac{\langle -5,1,1 \rangle}{3\sqrt{3}} = \frac{-5+2+3}{3\sqrt{3}} = 0$

10] min of

$f = x^2 + y^2 + z^2$ subj to $2x + 6y + 10z = 140 = g$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$2x = 2\lambda$ $x = \lambda$

$2y = 6\lambda$ $y = 3\lambda$

$2z = 10\lambda$ $z = 5\lambda$

$2x + 6y + 10z = 140$

$2\lambda + 18\lambda + 50\lambda = 140$

$70\lambda = 140$

$\lambda = 2$ $x = 2$, $y = 6$, $z = 10$

11] max & min of

$f = x^2 + y^2 - x - y + 1$ subj to $x^2 + y^2 \leq 1$

CRIT PTS $f_x = 2x - 1 = 0$, $f_y = 2y - 1 = 0$ $x = y = \frac{1}{2}$

$f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} + 1 = \frac{1}{2}$ ← min

Boundary $x^2 + y^2 = 1$

$2x - 1 = 2\lambda x \rightarrow 2x - 2\lambda x = 1 \Rightarrow x = \frac{1}{2-2\lambda}$

$2y - 1 = 2\lambda y$ $y = \frac{1}{2-2\lambda}$ $x = y$

$x^2 + y^2 = 1$

$2\left(\frac{1}{2-2\lambda}\right)^2 = 1 \rightarrow 2 - 2\lambda = \pm\sqrt{2}$

$\frac{2 \pm \sqrt{2}}{2} = \lambda = 1 \pm \frac{\sqrt{2}}{2}$ $x = \frac{1}{2 - 2(1 \pm \frac{\sqrt{2}}{2})}$

$= \frac{1}{2 - 2 \pm \sqrt{2}} = \frac{\pm 1}{\sqrt{2}}$

$y = \pm \frac{1}{\sqrt{2}}$

$f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2} + \frac{1}{2} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1$

$= 2 - \frac{2}{\sqrt{2}} = 2 - \sqrt{2}$

$f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 = 2 + \sqrt{2}$ ← max

min = 1/2

12. $\int_0^1 \int_{x^{1/2}}^{x^{1/3}} xy\,dy\,dx = \int_0^1 x\frac{y^2}{2}\Big|_{x^{1/2}}^{x^{1/3}} dy = \frac{1}{2}\int_0^1 x^{5/3} - x^2\,dx$

$= \frac{1}{2}\left[\frac{3}{8}x^{8/3} - \frac{x^3}{3}\right]_0^1$

$= \frac{1}{2}\left[\frac{3}{8} - \frac{1}{3}\right] = \left[\frac{3}{16} - \frac{1}{6}\right] = \frac{1}{48}$

Rev: $\int_0^1 \int_{y^3}^{y^2} xy\,dx\,dy = \int_0^1 \frac{x^2}{2}y\Big|_{y^3}^{y^2} dx = \frac{1}{2}\int_0^1 y^5 - y^7\,dy = \frac{1}{2}\left[\frac{y^6}{6} - \frac{y^8}{8}\right]_0^1 = \frac{1}{48}$

13) $\int_0^1 \int_{y^2}^{y} x^n + y^m\,dx\,dy = \int_0^1 \int_x^{\sqrt{x}} x^n + y^m\,dx\,dy$

The first avoids fractional powers, but both are straightforward

14) Mass $\int_{-1}^1 \int_{-1}^1 x^2y + 2x^2\,dy\,dx$

$= \int_{-1}^1 \frac{x^2y^2}{2} + 2x^2y\Big]_{-1}^1 dx = \int_{-1}^1 \cancel{\frac{x^2}{2}}^{\,0} + 4x^2\,dx = \left[\cancel{\frac{x^3}{6}} + \frac{4x^3}{3}\right]_{-1}^1 = \frac{8}{3}$

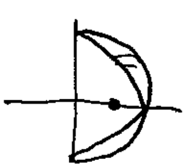
$\bar{x} = \frac{3}{8}\int_{-1}^1 \int_{-1}^1 x^3y + 2x^3\,dx\,dy = 0$ (symmetry)

$\bar{y} = \frac{3}{8}\int_{-1}^1 \int_{-1}^1 (x^2y^2 + 2x^2y)\,dy\,dx$

$= \frac{3}{8}\int_{-1}^1 \left[\frac{x^2y^3}{3} + \frac{2x^2y^2}{2}\right]_{-1}^1 dx$

$= \frac{3}{8}\int_{-1}^1 \frac{2x^2}{3}\,dx = \frac{3}{8}\cdot\frac{2x^3}{9}\Big|_{-1}^1 = 2\left[\frac{6}{72}\right] = \frac{1}{6}$

c.o.m = $(0, \frac{1}{6}$.

15) 

a) mass = $\int_{-1}^1 \int_{1-y^2}^{\sqrt{1-y^2}} x\,dx\,dy = \frac{1}{2}\int_{-1}^1 x^2\Big]_{1-y^2}^{\sqrt{1-y^2}} dy$

$= \frac{1}{2}\int_{-1}^1 (1-y^2) - (1-y^2)^2\,dy$

$= \frac{1}{2}\int_{-1}^1 y^2 - y^4\,dy = \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_{-1}^1$

$= \frac{1}{2}\frac{4}{15} = \frac{2}{15}$

b) $\bar{y} = 0$ by symmetry

c) $\bar{x} = \frac{15}{2}\cdot\int_{-1}^1 \int_{1-y^2}^{\sqrt{1-y^2}} x^2\,dx\,dy$

d) c.o.m is not on the plate. The plate is not convex (horseshoe shaped)

16 $\iiint_E \cos(x^2+y^2+z^2)^{3/2}\,dV$

$0 \le \rho \le 1$
$0 \le \theta \le \frac{\pi}{2}$
$0 \le \phi \le \pi/2$

$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \cos(\rho^3)\,\underline{\rho^2 \sin\phi}\; d\rho\, d\phi\, d\theta$

$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{3}\sin\rho^3\Big]_0^1 \sin\phi\; d\phi\, d\theta$

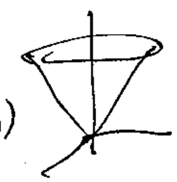
$= \int_0^{\pi/2} \left(\frac{1}{3}\sin 1\right)(-\cos\phi)\Big]_0^{\pi/2} d\theta$

$= \frac{1}{3}\sin 1 \cdot \frac{\pi}{2} = \frac{\pi}{6}\sin 1$

17. $y = \sin x$: $x = t$, $y = \sin t$, $0 \le t \le 2\pi$ Arclength : $\int_0^{2\pi}\sqrt{1+(\cos t)^2}\,dt$

$= \int_0^{2\pi}\sqrt{1+\cos^2 t}\,dt$

18.

a) 

$V = \int_{-h}^{h}\int_{-\sqrt{h^2-x^2}}^{\sqrt{h^2-x^2}} \left(h - \sqrt{x^2+y^2}\right) dy\, dx = \int_{-h}^{h}\int_{-\sqrt{h^2-x^2}}^{\sqrt{h^2-x^2}}\int_{\sqrt{x^2+y^2}}^{h} 1\; dz\, dy\, dx$

b) $\int_0^{2\pi}\int_0^h (h-r)\, r\, dr\, d\theta = \int_0^{2\pi} \frac{hr^2}{2} - \frac{r^3}{3}\Big]_0^h d\theta = \int_0^{2\pi} \frac{h^3}{6}\, dh$

$= \frac{1}{3}h^3\pi \quad \left(= \frac{1}{3}\pi R^2 h\right)$

c) sph $0 \le \rho \le \frac{h}{\cos\phi}$, $0 \le \theta < 2\pi$, $0 \le \phi \le \pi/4$

$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\frac{h}{\cos\phi}} \rho^2 \sin\phi\; d\rho\, d\theta\, d\phi$

$= \int_0^{\pi/4}\int_0^{2\pi} \frac{h^3}{3\cos^3\phi}\sin\phi\; d\theta\, d\phi = \int_0^{\pi/4} \frac{2\pi h^3}{3}\frac{\sin\phi}{\cos^3\phi}\, d\phi$ ← u-sub

$= \frac{\pi h^3}{3}$

19 $\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\,dV$

$0 \le \rho \le 1$
$0 \le \theta \le 2\pi$
$0 \le \phi \le \frac{\pi}{2}$

$$\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 e^{\rho^3}\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \frac{1}{3}(e-1)\,2\pi \,.$$

20. a) $\int_C x\,ds$ $\quad x = t$, $y = 2t^2 + 3$, $1 \le t \le 3$, $(ds = \sqrt{1+16t^2}$ $\rightarrow \int_1^3 t\sqrt{1+16t^2}\,dt$

$$= \frac{1}{32}\frac{2}{3}\left[1+16t^2\right]^{3/2}\Big|_1^3$$

$$= \frac{1}{48}\left[145^{3/2} - 17^{3/2}\right]$$

b) $\int_C \underset{P}{(x^2+2xy)}\,dx + \underset{Q}{(x^2+y^2)}\,dy$ $\quad \langle t^2\sin t, \cos t\rangle$, $0 \le t \le \pi/2$, $(0,1)$ to $(\frac{\pi^2}{4}, 0)$

$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$ ✓

$\langle P, Q\rangle = \vec{\nabla} f$ where $f = \frac{x^3}{3} + x^2 y + \frac{y^3}{3}$

so $\int P\,dx + Q\,dy = \frac{x^3}{3} + x^2 y + \frac{y^3}{3}\Big]_{(0,1)}^{(\frac{\pi^2}{4},0)}$

$$\rightarrow \frac{\pi^6}{192} - \frac{1}{3}$$

c)

$$\int_C (e^x + 2y)\,dx + (\arctan y + x^2)\,dy$$

$$= \int_0^2 \int_{x^2}^{2x} 2x - 2\,dy\,dx$$

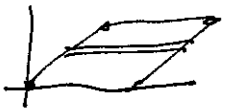
$$= \int_0^2 4x^2 - 4x - 2x^3 + 2x^2\,dx$$

$$= \int_0^2 6x^2 - 4x - 2x^3\,dx = 2x^3 - 2x^2 - \frac{x^4}{2}\Big]_0^2$$

$$= 16 - 8 - 8 = 0$$

d) $\oint_C (x^3 + 2y)\,dx + (x + \cos y)\,dy \rightarrow \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

($P = x^3 + 2y$, $Q = x + \cos y$)



$$= \int_0^2 \int_y^{y+4} 1 - 2 \, dx\,dy$$

$$= \int_0^2 -4 \, dy = -8$$

21 $\vec{F} = \langle 2y, 2x+y \rangle$

line segment:

$(-1,0)$ to $(1,0)$

$x = 2t$

$y = 0$

$0 \le t \le 1$

$$\int_C 2y\,dx + (2x+y)\,dy$$

$$= \int_0^1 0 + 0 = 0$$

halfcircle

$(-1,0)$ to $(1,0)$

$x = -\cos t$

$y = \sin t$

$0 \le t \le \pi$

$$\int_0^{\pi} \underbrace{2\sin^2 t}_{2y\,dx} + \underbrace{((-2\cos t) + \sin t)\cos t}_{(2x+y)\,dy}\, dt$$

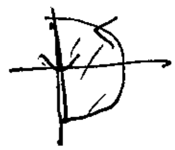
$$\int_0^{\pi} 2\sin^2 t - 2\cos^2 t + \sin t \cos t \, dt$$

$$= \int_0^{\pi} 2\cos 2t + \sin t \cos t \, dt$$

$$= \sin 2t + \frac{\sin^2 t}{2} \Big]_0^{\pi} = 0$$

Since $\frac{\partial Q}{\partial x} = 2 = \frac{\partial P}{\partial y}$, F is conservative, so $\int_C \vec{F} \cdot d\vec{r}$ is path independent

22. $\oint_C (\ln x + 2y)\,dx + (\sin(y)\, e^{\cos(y)} + x^2)\,dy$



Greens thm $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2$

$$\int_{-\pi/2}^{\pi/2} \int_0^1 (2r\cos\theta - 2)\, r \, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{2r^3}{3}\cos\theta - r^2 \Big|_0^1 = \int_{-\pi/2}^{\pi/2} 2/3 \cos\theta - 1 \, d\theta$$

$$= \frac{2}{3}\sin\theta - \theta \Big]_{-\pi/2}^{\pi/2}$$

$$= 4/3 - \pi = \boxed{4/3 - \pi}$$