Math 225: Final Review Questions

These are all questions from past final exams. They are representative of what you're likely to see on your final for this course. Pay careful attention to the last two, as they WILL be on the final.

1. Consider the triangle ABC where

$$A = (1, 1, 2), B = (2, 3, 4), C = (3, 4, 6)$$

   (a) Calculate $\mathbf{x} = \overrightarrow{AC}, \mathbf{y} = \overrightarrow{AB}$ use them to find the plane that contains the triangle ABC.
   (b) Find the area of triangle ABC (Hint: It's half of a certain parallelogram).
   (c) Calculate $\mathbf{z} = \text{proj}_{\mathbf{x}}\mathbf{y}$
   (d) Calculate $\frac{1}{2} \cdot |\mathbf{x}| \cdot |\mathbf{y} - \mathbf{z}|$ and explain why your answer makes sense. (Draw a picture)

2. Consider $A = (1, 2, 3), B = (-1, 0, 4), C = (2, -3, 5)$
   (a) Find the equations of the lines AB and AC and the angle between these lines.
   (b) Find the equation of the plane containing AB and AC.
   (c) Find the equation of the line normal to the plane passing through the point C.

3. Consider the circle of radius 1 centered at the point (1,0).
   (a) Find a parametrization for this circle
   (b) Find a polar equation for this circle
   (c) Verify (using any integral you wish) the circumference of this circle
   (d) Verify (using any integral you wish) the area of this circle.

4. Let $\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos(2t) \rangle$ describe the path of a particle in 3-space.
   (a) What are the permissible x, y and z values for $\mathbf{r}(t)$
   (b) Is this curve smooth? Why or why not?
   (c) This curve is the intersection of which two surfaces?

5. Find the tangent line to the parametric curve $\langle \ln t, 2\sqrt{t}, t^2 \rangle$ at the point $(0, 2, 1)$. Describe how to find the normal and osculating planes at this point.

6. Draw several level curves of the function $e^{\frac{y}{x}}$ (paying attention to spacing). Describe the surface these curves represent.

7. Let

$$f(x, y) = \frac{1}{x^2 + y^2 + 1}$$

   (a) Is f defined everywhere? Explain.
   (b) For which values of k does $f(x, y) = k$ produce level curves? What do these curves look like?

(c) Find the equation of the tangent plane to $f(x, y)$ at the point $(2, -1)$. Use it to approximate $f(1.97, -.97)$.

8. Find the tangent plane to the graph of the function $f(x, y) = e^x \cos(xy)$. Approximate $f(.01, .02)$.

9. Find the direction of maximum rate of change of the function $f(x, y, z) = \tan(x + 2y + 3z)$ at the point $(-5, 1, 1)$. Also, find the rate of change of f as we move towards the origin.

10. Find the minimum value of $f(x, y, z) = x^2+y^2+z^2$ subject to the constraint $2x+6y+10z = 140$

11. Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 - x - y + 1$ on the unit disk $x^2 + y^2 \leq 1$. (You'll be looking at points both inside and on the circle).

12. Compute
$$\int_0^1 \int_{\sqrt{x}}^{\sqrt[3]{x}} xy \, dy \, dx.$$
Then, reverse the order of integration and verify your answer.

13. Reverse the order of integration on
$$\int_0^1 \int_{y^2}^{y} x^n + y^m \, dx \, dy$$
Which integral would you rather compute?

14. A square plate bound by the lines $x = \pm 1$, $y = \pm 1$ has density equal to $\rho(x, y) = x^2 y + 2x^2$. Find the center of mass of this plate.

15. Consider a plate in the right half of the xy-plane bound by the curves $x^2 + y^2 = 1$ and $x = 1 - y^2$. The density of the plate is given by the function $\rho(x, y) = x$.

    (a) Find the mass of the plate.

    (b) Find $\overline{y}$ for the plate. (Hint: You may argue rather than compute here)

    (c) Set up the integral to compute $\overline{x}$. Do not attempt to compute your integral.

    (d) Is the center of mass actually on the plate? If not, explain how this can be.

16. Compute
$$\iiint_E \cos((x^2 + y^2 + z^2)^{\frac{3}{2}}) \, dV$$
where E is the portion of the unit sphere centered at the origin in the first octant.

17. Set up an integral to determine the length of one period of $y = \sin(x)$. Do not attempt to compute your integral.

18. Let V be the region bound by $z = \sqrt{x^2 + y^2}$ and $z = h$.

    (a) Set up the volume V as either a double or triple integral in rectangular coordinates.

    (b) Determine the volume of V by doing a double integral in polar coordinates. Your answer should be simple and verifiable from high school geometry.

(c) Verify your answer to part (b) by doing a triple integral in spherical coordinates. (Warning: May take some time).

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$$\iiint_E e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\ dV$$

where E half of the unit sphere above the xy-plane.

20. Compute the following integrals.

(a)

$$\int_C x\ ds$$

where C lies on the parabolic curve $y = 2x^2 + 3$ from $(1, 5)$ to $(3, 21)$.

(b)

$$\int_C (x^2 + 2xy)\ dx + (x^2 + y^2)\ dy$$

where C is the curve $\langle t^2 \sin(t), \cos(t) \rangle$ from $0 \leq t \leq \frac{\pi}{2}$

(c)

$$\oint_C (e^x + 2y)\ dx + (\arctan(y) + x^2)\ dx$$

where C is the parabolic segment along $y = x^2$ from $(0, 0)$ to $(2, 4)$ followed by the line segment from $(2, 4)$ to $(0, 0)$.

(d)

$$\oint_C (x^3 + 2y)\ dx + (x + \cos(y))\ dy$$

where C is the parallelogram with vertices $(0, 0), (2, 2), (6, 2)$, and $(4, 0)$, oriented counterclockwise.

21. Suppose that $\mathbf{F} = \langle 2y, 2x + y \rangle$. Check that $\int_C \mathbf{F} \cdot d\mathbf{r}$ for both the line segment from (-1, 0) to (1,0) and the semicircle between those two points. Why is this the case?

22. Compute $\oint_C \ln x + 2y\ dx + \sin(y)e^{\cos(y)} + x^2\ dy$, where C is the right half of the unit circle oriented counterclockwise.

23. List as many (at least 3) integral theorems as you can and draw any parallels that you can between them. Note: I want *theorems* and not *formulas* (ie, 'The integral for arclength' is not a *theorem*).

24. Choose a concept from single-variable calculus that was revisited in multivariable calculus and explain the parallels and the differences between the concept in the two contexts. Write your answer to someone who is finishing Calculus II and is thinking about taking Calculus III.