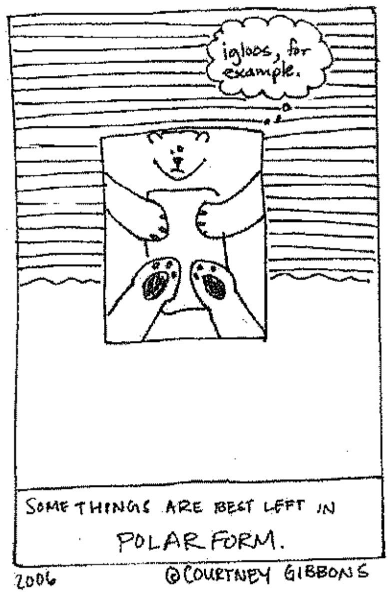
KEY

Math 225: Quiz the Eighth

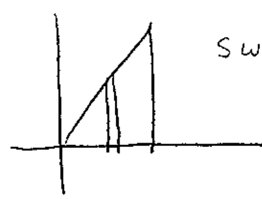
December 2, 2009

You know the drill by now. No books, no notes, no colleagues, and no answers without justification.
READ ALL QUESTIONS CAREFULLY



1. Find

$$\int_0^2 \int_{\frac{y}{2}}^1 \cos(x^2)\,dx\,dy \longrightarrow$$



switch the order

$x = y/2$

$y = 2x$

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$$\int_0^1 \int_0^{2x} \cos(x^2)\,dy\,dx$$

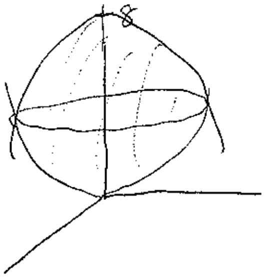
$$= \int_0^1 y\cos(x^2)\Big|_0^{2x} dx$$

$$= \int_0^1 2x\cos(x^2)\,dx = \sin(x^2)\Big|_0^1$$

(u-sub)

$$= \sin 1$$

2. Find the volume between the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$



Intersection:

$$x^2+y^2 = 8-x^2-y^2$$
$$2(x^2+y^2) = 8$$
$$x^2+y^2 = 4$$

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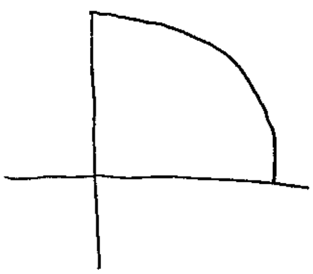
$$\iint_{x^2+y^2=4} (8-x^2-y^2)-(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^2 (8-2r^2)\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 8r - 2r^3\,dr\,d\theta$$
$$= \int_0^{2\pi} 4r^2 - \tfrac{1}{2}r^4 \Big]_0^2\,d\theta = 8\cdot 2\pi = \boxed{16\pi}$$

3. Find the center of mass of a plate in the shape of a unit quarter circle in the first quadrant if the density at any point is given by $\rho(x, y) = xy$. (You can use symmetry to save yourself some work here)



$$\text{Mass} = \iint_R \rho(x,y)\,dA = \iint_R xy\,dA$$

$$= \int_0^{\pi/2}\int_0^1 r^2\cos\theta\sin\theta\cdot r\,dr\,d\theta$$

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$$= \int_0^{\pi/2} \frac{r^4}{4}\Big|_0^1 \cos\theta\sin\theta\,d\theta = \frac{1}{4}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$
$$= \frac{1}{4}\,\frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = \frac{1}{8}$$

$$\bar{x} = \frac{M_y}{\text{mass}} = \frac{\iint_R x\rho(x,y)\,dA}{1/8}$$

$$= \frac{\int_0^{\pi/2}\int_0^1 r^3\cos^2\theta\sin\theta\,r\,dr\,d\theta}{1/8} = \frac{\int_0^{\pi/2} \frac{r^5}{5}\Big|_0^1 \cos^2\theta\sin\theta\,d\theta}{1/8}$$

$$= -\frac{1}{5}\left[\frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{1}{15}\Big/\frac{1}{8} = 8/15$$

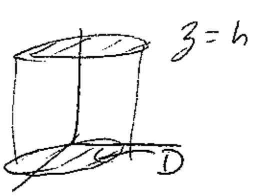
$\bar{y} = \bar{x}$ since plate, density are symmetric

Thus $(\bar{x},\bar{y}) = \left(\frac{8}{15}, \frac{8}{15}\right)$

4. (a) Determine the volume of a cylinder of radius R and height h. Prove your answer is correct by solving the appropriate integral.

$V = \pi R^2 h$ from High School Geometry

Calc III



$$\iint_D h \, dA = \int_0^{2\pi}\int_0^R h r \, dr \, d\theta$$
$$= \int_0^{2\pi} \frac{hr^2}{2}\Big|_0^R d\theta = \int_0^{2\pi} \frac{hR^2}{2} d\theta = \pi h R^2$$

✓, ✓, etc.

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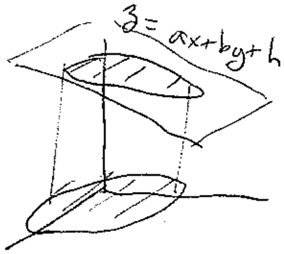
(b) Find the volume bound by the cylinder $x^2 + y^2 = R^2$, the xy-plane, and the plane $z = ax + by + h$ and comment on your answer. (This may help you with part (a). Assume that h is large enough so that the slanted plane lies entirely above the $x - y$ plane inside the cylinder.).



$$\iint_D ax+by+h \, dA$$
$$= \int_0^{2\pi}\int_0^R ar^2\cos\theta + br^2\sin\theta + hr \, dr \, d\theta$$
$$= \int_0^{2\pi} \left[\frac{ar^3}{3}\cos\theta + \frac{br^3}{3}\sin\theta + \frac{hr^2}{2} \right]_0^R d\theta$$
$$= \int_0^{2\pi} \frac{aR^3}{3}\cos\theta + \frac{bR^3}{3}\sin\theta + \frac{hR^2}{2} \, d\theta$$
$$= \underbrace{\frac{aR^3}{3}\sin\theta}_{0} - \underbrace{\frac{bR^3}{3}\cos\theta}_{0} + \frac{hR^2}{2}\theta \Big]_0^{2\pi}$$
$$= \frac{hR^2}{2}(2\pi) = \pi h R^2$$

The volumes are the same! The "tilt" adds and subtracts equal values from the cylinder.

5. Find the area of the surface $z = \frac{2}{3}(x^{\frac{3}{2}} + y^{\frac{3}{2}})$ above the rectangle $[0,1] \times [0,1]$.

$$f_x = \frac{3}{2} \cdot \frac{2}{3} x^{\frac{1}{2}} = x^{\frac{1}{2}}$$

$$f_y = \frac{3}{2} \cdot \frac{2}{3} y^{\frac{1}{2}} = y^{\frac{1}{2}}$$

$$S.A. = \int_0^1 \int_0^1 \sqrt{1 + x + y}\, dy\, dx$$

$$= \int_0^1 \frac{2}{3}(1+x+y)^{3/2} \Big|_0^1 dx$$

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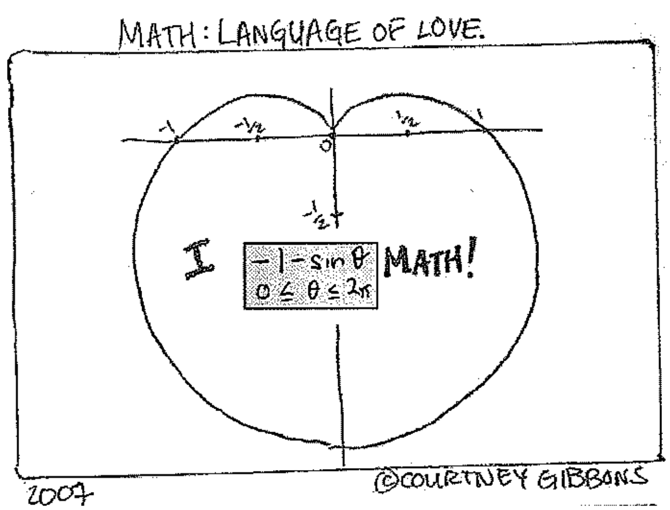
$$= \int_0^1 \frac{2}{3}\left[(2+x)^{3/2} - (1+x)^{3/2}\right] dx$$

$$= \frac{2}{3}\left[\frac{2}{5}(2+x)^{5/2} - \frac{2}{5}(1+x)^{5/2}\right]_0^1$$

$$= \frac{4}{15}\left(3^{5/2} - 2^{5/2} - 2^{5/2} + 1^{5/2}\right)$$

6. (Bonus) Fill in the grey box below.



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