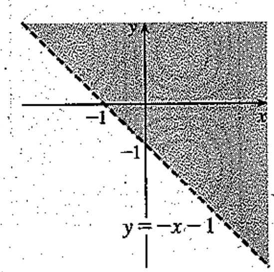
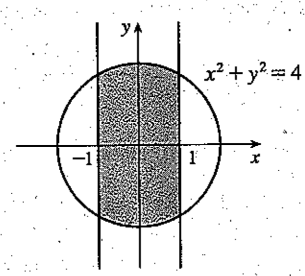
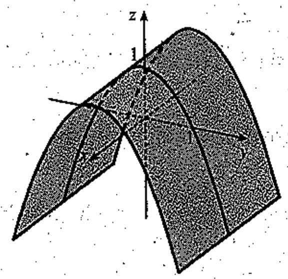
1. $\ln(x+y+1)$ is defined only when $x+y+1>0 \quad \Leftrightarrow \quad y>-x-1$, so the domain of f is $\{(x,y) \mid y>-x-1\}$, all those points above the line $y=-x-1$.



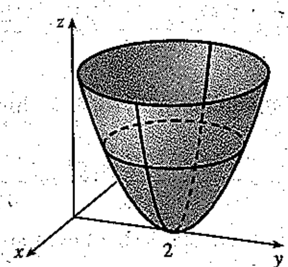
2. $\sqrt{4-x^2-y^2}$ is defined only when $4-x^2-y^2 \geq 0 \quad \Leftrightarrow \quad x^2+y^2 \leq 4$, and $\sqrt{1-x^2}$ is defined only when $1-x^2 \geq 0 \quad \Leftrightarrow \quad -1 \leq x \leq 1$, so the domain of f is $\left\{(x,y) \mid -1 \leq x \leq 1, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\right\}$, which consists of those points on or inside the circle $x^2+y^2=4$ for $-1 \leq x \leq 1$.



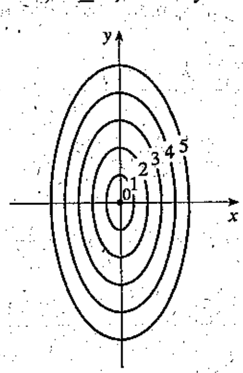
3. $z=f(x,y)=1-y^2$, a parabolic cylinder



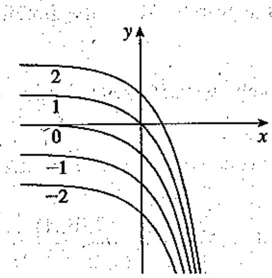
4. $z=f(x,y)=x^2+(y-2)^2$, a circular paraboloid with vertex $(0,2,0)$ and axis parallel to the z-axis



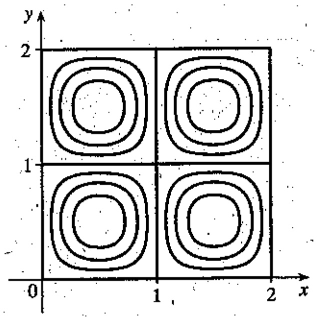
5. The level curves are $\sqrt{4x^2+y^2}=k$ or $4x^2+y^2=k^2$, $k \geq 0$, a family of ellipses.



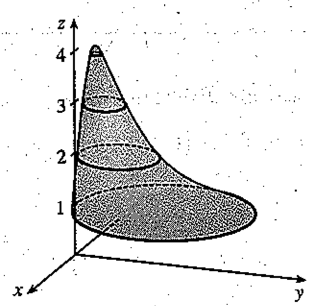
6. The level curves are $e^x+y=k$ or $y=-e^x+k$, a family of exponential curves.



7.



8.



9. f is a rational function, so it is continuous on its domain. Since f is defined at $(1,1)$, we use direct substitution to evaluate the limit: $\displaystyle\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+2y^2} = \frac{2(1)(1)}{1^2+2(1)^2} = \frac{2}{3}$.

10. As $(x,y) \to (0,0)$ along the x-axis, $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$ along this line. But $f(x,x) = 2x^2/(3x^2) = \frac{2}{3}$, so as $(x,y) \to (0,0)$ along the line $x = y$, $f(x,y) \to \frac{2}{3}$. Thus the limit doesn't exist.

11. (a) $T_x(6,4) = \displaystyle\lim_{h\to 0} \frac{T(6+h,4) - T(6,4)}{h}$, so we can approximate $T_x(6,4)$ by considering $h = \pm 2$ and using the values given in the table: $T_x(6,4) \approx \dfrac{T(8,4) - T(6,4)}{2} = \dfrac{86-80}{2} = 3$,

$T_x(6,4) \approx \dfrac{T(4,4) - T(6,4)}{-2} = \dfrac{72-80}{-2} = 4$. Averaging these values, we estimate $T_x(6,4)$ to be approximately $3.5°\text{C/m}$. Similarly, $T_y(6,4) = \displaystyle\lim_{h\to 0} \frac{T(6,4+h) - T(6,4)}{h}$, which we can approximate with $h = \pm 2$:

$T_y(6,4) \approx \dfrac{T(6,6) - T(6,4)}{2} = \dfrac{75-80}{2} = -2.5$, $T_y(6,4) \approx \dfrac{T(6,2) - T(6,4)}{-2} = \dfrac{87-80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6,4)$ to be approximately $-3.0°\text{C/m}$.

(b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 14.6.9, $D_{\mathbf{u}}\,T(6,4) = \nabla T(6,4) \cdot \mathbf{u} = T_x(6,4)\,\frac{1}{\sqrt{2}} + T_y(6,4)\,\frac{1}{\sqrt{2}}$. Using our estimates from part (a), we have $D_{\mathbf{u}}\,T(6,4) \approx (3.5)\,\frac{1}{\sqrt{2}} + (-3.0)\,\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6,4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately $0.35°\text{C/m}$.

Alternatively, we can use Definition 14.6.2: $D_{\mathbf{u}}\,T(6,4) = \displaystyle\lim_{h\to 0} \frac{T\!\left(6 + h\,\frac{1}{\sqrt{2}}, 4 + h\,\frac{1}{\sqrt{2}}\right) - T(6,4)}{h}$,

which we can estimate with $h = \pm 2\sqrt{2}$. Then $D_{\mathbf{u}}\,T(6,4) \approx \dfrac{T(8,6) - T(6,4)}{2\sqrt{2}} = \dfrac{80-80}{2\sqrt{2}} = 0$,

$D_{\mathbf{u}}\,T(6,4) \approx \dfrac{T(4,2) - T(6,4)}{-2\sqrt{2}} = \dfrac{74-80}{-2\sqrt{2}} = \dfrac{3}{\sqrt{2}}$. Averaging these values, we have $D_{\mathbf{u}}\,T(6,4) \approx \frac{3}{2\sqrt{2}} \approx 1.1°\text{C/m}$.

(c) $T_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[T_x(x,y)] = \displaystyle\lim_{h\to 0} \frac{T_x(x,y+h) - T_x(x,y)}{h}$, so $T_{xy}(6,4) = \displaystyle\lim_{h\to 0} \frac{T_x(6,4+h) - T_x(6,4)}{h}$ which we can estimate with $h = \pm 2$. We have $T_x(6,4) \approx 3.5$ from part (a), but we will also need values for $T_x(6,6)$ and $T_x(6,2)$. If we use $h = \pm 2$ and the values given in the table, we have

$$T_x(6,6) \approx \frac{T(8,6) - T(6,6)}{2} = \frac{80 - 75}{2} = 2.5, \; T_x(6,6) \approx \frac{T(4,6) - T(6,6)}{-2} = \frac{68 - 75}{-2} = 3.5.$$

Averaging these values, we estimate $T_x(6,6) \approx 3.0$. Similarly,

$$T_x(6,2) \approx \frac{T(8,2) - T_x(6,2)}{2} = \frac{90 - 87}{2} = 1.5, \; T_x(6,2) \approx \frac{T(4,2) - T(6,2)}{-2} = \frac{74 - 87}{-2} = 6.5.$$

Averaging these values, we estimate $T_x(6,2) \approx 4.0$. Finally, we estimate $T_{xy}(6,4)$:

$$T_{xy}(6,4) \approx \frac{T_x(6,6) - T_x(6,4)}{2} = \frac{3.0 - 3.5}{2} = -0.25, \; T_{xy}(6,4) \approx \frac{T_x(6,2) - T_x(6,4)}{-2} = \frac{4.0 - 3.5}{-2} = -0.25.$$

Averaging these values, we have $T_{xy}\,(6,4) \approx -0.25$.

12. From the table, $T(6,4) = 80$, and from Exercise 11 we estimated $T_x(6,4) \approx 3.5$ and $T_y(6,4) \approx -3.0$. The linear approximation then is

$$T\,(x,y) \approx T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4) \approx 80 + 3.5(x-6) - 3(y-4) = 3.5x - 3y + 71$$

Thus at the point $(5, 3.8)$, we can use the linear approximation to estimate $T(5, 3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1°\text{C}$.

13. $f(x,y) = (5y^3 + 2x^2y)^8 \quad \Rightarrow \quad f_x = 8(5y^3 + 2x^2y)^7(4xy) = 32xy(5y^3 + 2x^2y)^7$,

$f_y = 8(5y^3 + 2x^2y)^7(15y^2 + 2x^2) = (16x^2 + 120y^2)(5y^3 + 2x^2y)^7$

14. $g(u,v) = \dfrac{u + 2v}{u^2 + v^2} \quad \Rightarrow \quad g_u = \dfrac{(u^2 + v^2)(1) - (u + 2v)(2u)}{(u^2 + v^2)^2} = \dfrac{v^2 - u^2 - 4uv}{(u^2 + v^2)^2}$,

$g_v = \dfrac{(u^2 + v^2)(2) - (u + 2v)(2v)}{(u^2 + v^2)^2} = \dfrac{2u^2 - 2v^2 - 2uv}{(u^2 + v^2)^2}$

15. $F(\alpha,\beta) = \alpha^2 \ln(\alpha^2 + \beta^2) \quad \Rightarrow \quad F_\alpha = \alpha^2 \cdot \dfrac{1}{\alpha^2 + \beta^2}\,(2\alpha) + \ln(\alpha^2 + \beta^2) \cdot 2\alpha = \dfrac{2\alpha^3}{\alpha^2 + \beta^2} + 2\alpha \ln(\alpha^2 + \beta^2)$,

$F_\beta = \alpha^2 \cdot \dfrac{1}{\alpha^2 + \beta^2}\,(2\beta) = \dfrac{2\alpha^2\beta}{\alpha^2 + \beta^2}$

16. $G(x,y,z) = e^{xz} \sin(y/z) \quad \Rightarrow \quad G_x = ze^{xz} \sin(y/z),\; G_y = e^{xz} \cos(y/z)(1/z) = (e^{xz}/z) \cos(y/z)$,

$G_z = e^{xz} \cdot \cos(y/z)(-y/z^2) + \sin(y/z) \cdot xe^{xz} = e^{xz}\left[x \sin(y/z) - (y/z^2) \cos(y/z)\right]$

17. $S(u,v,w) = u \arctan(v\sqrt{w}) \quad \Rightarrow \quad S_u = \arctan(v\sqrt{w}),\; S_v = u \cdot \dfrac{1}{1 + (v\sqrt{w})^2}\,(\sqrt{w}) = \dfrac{u\sqrt{w}}{1 + v^2w}$,

$S_w = u \cdot \dfrac{1}{1 + (v\sqrt{w})^2}\left(v \cdot \tfrac{1}{2}w^{-1/2}\right) = \dfrac{uv}{2\sqrt{w}\,(1 + v^2w)}$

18. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D \quad \Rightarrow$

$\partial C/\partial T = 4.6 - 0.11T + 0.00087T^2 - 0.01(S - 35)$, $\partial C/\partial S = 1.34 - 0.01T$, and $\partial C/\partial D = 0.016$. When $T = 10$, $S = 35$, and $D = 100$ we have $\partial C/\partial T = 4.6 - 0.11(10) + 0.00087(10)^2 - 0.01(35 - 35) \approx 3.587$, thus in 10°C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree

Celsius that the water temperature rises. Similarly, $\partial C/\partial S = 1.34 - 0.01(10) = 1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C/\partial D = 0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

19. $f(x,y) = 4x^3 - xy^2 \Rightarrow f_x = 12x^2 - y^2,\ f_y = -2xy,\ f_{xx} = 24x,\ f_{yy} = -2x,\ f_{xy} = f_{yx} = -2y$

20. $z = xe^{-2y} \Rightarrow z_x = e^{-2y},\ z_y = -2xe^{-2y},\ z_{xx} = 0,\ z_{yy} = 4xe^{-2y},\ z_{xy} = z_{yx} = -2e^{-2y}$

21. $f(x,y,z) = x^k y^l z^m \Rightarrow f_x = kx^{k-1}y^l z^m,\ f_y = lx^k y^{l-1} z^m,\ f_z = mx^k y^l z^{m-1},\ f_{xx} = k(k-1)x^{k-2}y^l z^m,$

$f_{yy} = l(l-1)x^k y^{l-2} z^m,\ f_{zz} = m(m-1)x^k y^l z^{m-2},\ f_{xy} = f_{yx} = klx^{k-1}y^{l-1}z^m,\ f_{xz} = f_{zx} = kmx^{k-1}y^l z^{m-1},$

$f_{yz} = f_{zy} = lmx^k y^{l-1} z^{m-1}$

22. $v = r\cos(s+2t) \Rightarrow v_r = \cos(s+2t),\ v_s = -r\sin(s+2t),\ v_t = -2r\sin(s+2t),\ v_{rr} = 0,\ v_{ss} = -r\cos(s+2t),$

$v_{tt} = -4r\cos(s+2t),\ v_{rs} = v_{sr} = -\sin(s+2t),\ v_{rt} = v_{tr} = -2\sin(s+2t),\ v_{st} = v_{ts} = -2r\cos(s+2t)$

23. $z = xy + xe^{y/x} \Rightarrow \dfrac{\partial z}{\partial x} = y - \dfrac{y}{x}e^{y/x} + e^{y/x},\ \dfrac{\partial z}{\partial y} = x + e^{y/x}$ and

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(y - \frac{y}{x}e^{y/x} + e^{y/x}\right) + y\left(x + e^{y/x}\right) = xy - ye^{y/x} + xe^{y/x} + xy + ye^{y/x} = xy + xy + xe^{y/x} = xy + z.$$

24. $z = \sin(x + \sin t) \Rightarrow \dfrac{\partial z}{\partial x} = \cos(x + \sin t),\ \dfrac{\partial z}{\partial t} = \cos(x + \sin t)\cos t,$

$\dfrac{\partial^2 z}{\partial x \partial t} = -\sin(x+\sin t)\cos t,\ \dfrac{\partial^2 z}{\partial x^2} = -\sin(x + \sin t)$ and

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x \partial t} = \cos(x+\sin t)\left[-\sin(x+\sin t)\cos t\right] = \cos(x+\sin t)(\cos t)\left[-\sin(x+\sin t)\right] = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}.$$

25. (a) $z_x = 6x + 2 \Rightarrow z_x(1,-2) = 8$ and $z_y = -2y \Rightarrow z_y(1,-2) = 4$, so an equation of the tangent plane is $z - 1 = 8(x-1) + 4(y+2)$ or $z = 8x + 4y + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(1,-2,1)$ is $\langle 8, 4, -1\rangle$. Then parametric equations for the normal line there are $x = 1 + 8t,\ y = -2 + 4t,\ z = 1 - t$, and symmetric equations are $\dfrac{x-1}{8} = \dfrac{y+2}{4} = \dfrac{z-1}{-1}$.

26. (a) $z_x = e^x \cos y \Rightarrow z_x(0,0) = 1$ and $z_y = -e^x \sin y \Rightarrow z_y(0,0) = 0$, so an equation of the tangent plane is $z - 1 = 1(x-0) + 0(y-0)$ or $z = x + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(0,0,1)$ is $\langle 1, 0, -1\rangle$. Then parametric equations for the normal line there are $x = t,\ y = 0,\ z = 1 - t$, and symmetric equations are $x = 1 - z,\ y = 0$.

27. (a) Let $F(x,y,z) = x^2 + 2y^2 - 3z^2$. Then $F_x = 2x,\ F_y = 4y,\ F_z = -6z$, so $F_x(2,-1,1) = 4,\ F_y(2,-1,1) = -4,$ $F_z(2,-1,1) = -6$. From Equation 14.6.19, an equation of the tangent plane is $4(x-2) - 4(y+1) - 6(z-1) = 0$ or, equivalently, $2x - 2y - 3z = 3$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x-2}{4} = \dfrac{y+1}{-4} = \dfrac{z-1}{-6}$.

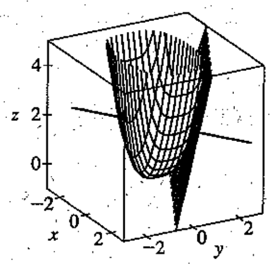
28. (a) Let $F(x,y,z) = xy + yz + zx$. Then $F_x = y + z$, $F_y = x + z$, $F_z = x + y$, so $F_x(1,1,1) = F_y(1,1,1) = F_z(1,1,1) = 2$. From Equation 14.6.19, an equation of the tangent plane is $2(x-1) + 2(y-1) + 2(z-1) = 0$ or, equivalently, $x + y + z = 3$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x-1}{2} = \dfrac{y-1}{2} = \dfrac{z-1}{2}$ or, equivalently, $x = y = z$.

29. (a) Let $F(x,y,z) = x + 2y + 3z - \sin(xyz)$. Then $F_x = 1 - yz\cos(xyz)$, $F_y = 2 - xz\cos(xyz)$, $F_z = 3 - xy\cos(xyz)$, so $F_x(2,-1,0) = 1$, $F_y(2,-1,0) = 2$, $F_z(2,-1,0) = 5$. From Equation 14.6.19, an equation of the tangent plane is $1(x-2) + 2(y+1) + 5(z-0) = 0$ or $x + 2y + 5z = 0$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z}{5}$ or $x - 2 = \dfrac{y+1}{2} = \dfrac{z}{5}$. Parametric equations are $x = 2 + t$, $y = -1 + 2t$, $z = 5t$.

30. Let $f(x,y) = x^2 + y^4$. Then $f_x(x,y) = 2x$ and $f_y(x,y) = 4y^3$, so $f_x(1,1) = 2$, $f_y(1,1) = 4$ and an equation of the tangent plane is $z - 2 = 2(x-1) + 4(y-1)$ or $2x + 4y - z = 4$. A normal vector to the tangent plane is $\langle 2, 4, -1\rangle$ so the normal line is given by $\dfrac{x-1}{2} = \dfrac{y-1}{4} = \dfrac{z-2}{-1}$ or $x = 1 + 2t$, $y = 1 + 4t$, $z = 2 - t$.



31. The hyperboloid is a level surface of the function $F(x,y,z) = x^2 + 4y^2 - z^2$, so a normal vector to the surface at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 8y_0, -2z_0\rangle$. A normal vector for the plane $2x + 2y + z = 5$ is $\langle 2, 2, 1\rangle$. For the planes to be parallel, we need the normal vectors to be parallel, so $\langle 2x_0, 8y_0, -2z_0\rangle = k\,\langle 2, 2, 1\rangle$, or $x_0 = k$, $y_0 = \frac{1}{4}k$, and $z_0 = -\frac{1}{2}k$. But $x_0^2 + 4y_0^2 - z_0^2 = 4 \;\Rightarrow\; k^2 + \frac{1}{4}k^2 - \frac{1}{4}k^2 = 4 \;\Rightarrow\; k^2 = 4 \;\Rightarrow\; k = \pm 2$. So there are two such points: $\left(2, \frac{1}{2}, -1\right)$ and $\left(-2, -\frac{1}{2}, 1\right)$.

32. $u = \ln(1 + se^{2t}) \;\Rightarrow\; du = \dfrac{\partial u}{\partial s}\,ds + \dfrac{\partial u}{\partial t}\,dt = \dfrac{e^{2t}}{1 + se^{2t}}\,ds + \dfrac{2se^{2t}}{1 + se^{2t}}\,dt$

33. $f(x,y,z) = x^3\sqrt{y^2 + z^2} \;\Rightarrow\; f_x(x,y,z) = 3x^2\sqrt{y^2 + z^2}$, $f_y(x,y,z) = \dfrac{yx^3}{\sqrt{y^2 + z^2}}$, $f_z(x,y,z) = \dfrac{zx^3}{\sqrt{y^2 + z^2}}$, so $f(2,3,4) = 8(5) = 40$, $f_x(2,3,4) = 3(4)\sqrt{25} = 60$, $f_y(2,3,4) = \frac{3(8)}{\sqrt{25}} = \frac{24}{5}$, and $f_z(2,3,4) = \frac{4(8)}{\sqrt{25}} = \frac{32}{5}$. Then the linear approximation of f at $(2,3,4)$ is

$$\begin{aligned} f(x,y,z) &\approx f(2,3,4) + f_x(2,3,4)(x-2) + f_y(2,3,4)(y-3) + f_z(2,3,4)(z-4) \\ &= 40 + 60(x-2) + \tfrac{24}{5}(y-3) + \tfrac{32}{5}(z-4) = 60x + \tfrac{24}{5}y + \tfrac{32}{5}z - 120 \end{aligned}$$

Then $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2} = f(1.98, 3.01, 3.97) \approx 60(1.98) + \frac{24}{5}(3.01) + \frac{32}{5}(3.97) - 120 = 38.656$.

34. (a) $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the calculated area is about $dA = 6(0.002) + \frac{5}{2}(0.002) = 0.017\text{ m}^2$ or 170 cm^2.

(b) $z = \sqrt{x^2+y^2}$, $dz = \dfrac{x}{\sqrt{x^2+y^2}}\,dx + \dfrac{y}{\sqrt{x^2+y^2}}\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the calculated hypotenuse length is about $dz = \frac{5}{13}(0.002) + \frac{12}{13}(0.002) = \frac{0.17}{65} \approx 0.0026$ m or 0.26 cm.

35. $\dfrac{du}{dp} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dp} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dp} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dp} = 2xy^3(1+6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$

36. $\dfrac{\partial v}{\partial s} = \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial s} = \left(2x\sin y + y^2e^{xy}\right)(1) + \left(x^2\cos y + xye^{xy} + e^{xy}\right)(t)$.

$s=0, t=1 \Rightarrow x=2, y=0$, so $\dfrac{\partial v}{\partial s} = 0 + (4+1)(1) = 5$.

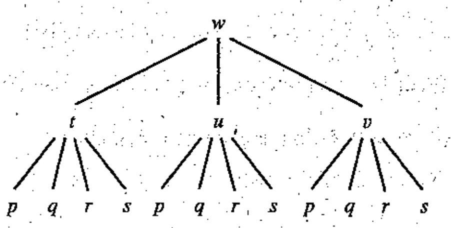
$\dfrac{\partial v}{\partial t} = \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial t} = \left(2x\sin y + y^2e^{xy}\right)(2) + \left(x^2\cos y + xye^{xy} + e^{xy}\right)(s) = 0 + 0 = 0$.

37. By the Chain Rule, $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$. When $s=1$ and $t=2$, $x = g(1,2) = 3$ and $y = h(1,2) = 6$, so

$\dfrac{\partial z}{\partial s} = f_x(3,6)g_s(1,2) + f_y(3,6)\,h_s(1,2) = (7)(-1) + (8)(-5) = -47$. Similarly, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$, so

$\dfrac{\partial z}{\partial t} = f_x(3,6)g_t(1,2) + f_y(3,6)\,h_t(1,2) = (7)(4) + (8)(10) = 108$.

38.



Using the tree diagram as a guide, we have

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial p} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p} \qquad \frac{\partial w}{\partial q} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial q}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$

39. $\dfrac{\partial z}{\partial x} = 2xf'(x^2-y^2)$, $\dfrac{\partial z}{\partial y} = 1 - 2yf'(x^2-y^2)$ $\left[\text{where } f' = \dfrac{df}{d(x^2-y^2)}\right]$. Then

$y\,\dfrac{\partial z}{\partial x} + x\,\dfrac{\partial z}{\partial y} = 2xyf'(x^2-y^2) + x - 2xyf'(x^2-y^2) = x$.

40. $A = \frac{1}{2}xy\sin\theta$, $dx/dt = 3$, $dy/dt = -2$, $d\theta/dt = 0.05$, and $\dfrac{dA}{dt} = \dfrac{1}{2}\left[(y\sin\theta)\,\dfrac{dx}{dt} + (x\sin\theta)\,\dfrac{dy}{dt} + (xy\cos\theta)\,\dfrac{d\theta}{dt}\right]$.

So when $x = 40$, $y = 50$ and $\theta = \frac{\pi}{6}$, $\dfrac{dA}{dt} = \dfrac{1}{2}\left[(25)(3) + (20)(-2) + \left(1000\sqrt{3}\right)(0.05)\right] = \dfrac{35 + 50\sqrt{3}}{2} \approx 60.8\text{ in}^2\text{/s}$.

41. $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\,y + \dfrac{\partial z}{\partial v}\,\dfrac{-y}{x^2}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= y\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{2y}{x^3}\frac{\partial z}{\partial v} + \frac{-y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = \frac{2y}{x^3}\frac{\partial z}{\partial v} + y\left(\frac{\partial^2 z}{\partial u^2}\,y + \frac{\partial^2 z}{\partial v\,\partial u}\,\frac{-y}{x^2}\right) + \frac{-y}{x^2}\left(\frac{\partial^2 z}{\partial v^2}\,\frac{-y}{x^2} + \frac{\partial^2 z}{\partial u\,\partial v}\,y\right)\\ &= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y^2\,\frac{\partial^2 z}{\partial u^2} - \frac{2y^2}{x^2}\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Also $\dfrac{\partial z}{\partial y} = x\,\dfrac{\partial z}{\partial u} + \dfrac{1}{x}\dfrac{\partial z}{\partial v}$ and

$$\frac{\partial^2 z}{\partial y^2} = x\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = x\left(\frac{\partial^2 z}{\partial u^2}x + \frac{\partial^2 z}{\partial v\,\partial u}\,\frac{1}{x}\right) + \frac{1}{x}\left(\frac{\partial^2 z}{\partial v^2}\,\frac{1}{x} + \frac{\partial^2 z}{\partial u\,\partial v}\,x\right) = x^2\,\frac{\partial^2 z}{\partial u^2} + 2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}$$

Thus

$$\begin{aligned}x^2\,\frac{\partial^2 z}{\partial x^2} - y^2\,\frac{\partial^2 z}{\partial y^2} &= \frac{2y}{x}\frac{\partial z}{\partial v} + x^2y^2\,\frac{\partial^2 z}{\partial u^2} - 2y^2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2} - x^2y^2\,\frac{\partial^2 z}{\partial u^2} - 2y^2\,\frac{\partial^2 z}{\partial u\,\partial v} - \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}\\ &= \frac{2y}{x}\frac{\partial z}{\partial v} - 4y^2\,\frac{\partial^2 z}{\partial u\,\partial v} = 2v\,\frac{\partial z}{\partial v} - 4uv\,\frac{\partial^2 z}{\partial u\,\partial v}\end{aligned}$$

since $y = xv = \dfrac{uv}{y}$ or $y^2 = uv$.

42. $\cos(xyz) = 1 + x^2y^2 + z^2$, so let $F(x,y,z) = 1 + x^2y^2 + z^2 - \cos(xyz) = 0$. Then by Equations 14.5.7 we have $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2xy^2 + \sin(xyz)\cdot yz}{2z + \sin(xyz)\cdot xy} = -\dfrac{2xy^2 + yz\sin(xyz)}{2z + xy\sin(xyz)}$,

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{2x^2y + \sin(xyz)\cdot xz}{2z + \sin(xyz)\cdot xy} = -\dfrac{2x^2y + xz\sin(xyz)}{2z + xy\sin(xyz)}$.

43. $f(x,y,z) = x^2e^{yz^2} \quad\Rightarrow\quad \nabla f = \langle f_x, f_y, f_z\rangle = \left\langle 2xe^{yz^2}, x^2e^{yz^2}\cdot z^2, x^2e^{yz^2}\cdot 2yz\right\rangle = \left\langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2}\right\rangle$

44. (a) By Theorem 14.6.15, the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as the gradient vector.

(b) It is a minimum when $\mathbf{u}$ is in the direction opposite to that of the gradient vector (that is, $\mathbf{u}$ is in the direction of $-\nabla f$), since $D_{\mathbf{u}}\,f = |\nabla f|\cos\theta$ (see the proof of Theorem 14.6.15) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when $\mathbf{u}$ is perpendicular to the gradient vector, since then $D_{\mathbf{u}}\,f = \nabla f\cdot\mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}}\,f = |\nabla f|\cos\theta = \frac{1}{2}\,|\nabla f| \quad\Leftrightarrow\quad \cos\theta = \frac{1}{2} \quad\Leftrightarrow\quad \theta = \frac{\pi}{3}$.

45. $f(x,y) = x^2e^{-y} \quad\Rightarrow\quad \nabla f = \langle 2xe^{-y}, -x^2e^{-y}\rangle$, $\nabla f(-2,0) = \langle -4,-4\rangle$. The direction is given by $\langle 4,-3\rangle$, so $\mathbf{u} = \frac{1}{\sqrt{4^2+(-3)^2}}\langle 4,-3\rangle = \frac{1}{5}\langle 4,-3\rangle$ and $D_{\mathbf{u}}\,f(-2,0) = \nabla f(-2,0)\cdot\mathbf{u} = \langle -4,-4\rangle\cdot\frac{1}{5}\langle 4,-3\rangle = \frac{1}{5}(-16+12) = -\frac{4}{5}$.

46. $\nabla f = \left\langle 2xy + \sqrt{1+z}, x^2, x/\left(2\sqrt{1+z}\right)\right\rangle$, $\nabla f(1,2,3) = \left\langle 6, 1, \frac{1}{4}\right\rangle$, $\mathbf{u} = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right\rangle$. Then $D_{\mathbf{u}}\,f(1,2,3) = \frac{25}{6}$.

47. $\nabla f = \langle 2xy, x^2 + 1/(2\sqrt{y})\rangle$, $|\nabla f(2,1)| = |\langle 4, \frac{9}{2}\rangle|$. Thus the maximum rate of change of f at $(2,1)$ is $\frac{\sqrt{145}}{2}$ in the direction $\langle 4, \frac{9}{2}\rangle$.

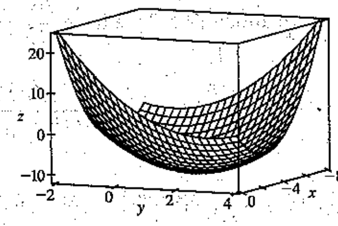
48. $\nabla f = \langle zye^{xy}, zxe^{xy}, e^{xy}\rangle$, $\nabla f(0,1,2) = \langle 2,0,1\rangle$ is the direction of most rapid increase while the rate is $|\langle 2,0,1\rangle| = \sqrt{5}$.

49. First we draw a line passing through Homestead and the eye of the hurricane. We can approximate the directional derivative at Homestead in the direction of the eye of the hurricane by the average rate of change of wind speed between the points where this line intersects the contour lines closest to Homestead. In the direction of the eye of the hurricane, the wind speed changes from 45 to 50 knots. We estimate the distance between these two points to be approximately 8 miles, so the rate of change of wind speed in the direction given is approximately $\frac{50-45}{8} = \frac{5}{8} = 0.625$ knot/mi.

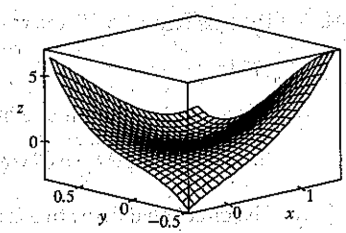
50. The surfaces are $f(x,y,z) = z - 2x^2 + y^2 = 0$ and $g(x,y,z) = z - 4 = 0$. The tangent line is perpendicular to both ∇f and ∇g at $(-2,2,4)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ is therefore parallel to the line. $\nabla f(x,y,z) = \langle -4x, 2y, 1\rangle \Rightarrow$ $\nabla f(-2,2,4) = \langle 8,4,1\rangle$, $\nabla g(x,y,z) = \langle 0,0,1\rangle \Rightarrow \nabla g\langle -2,2,4\rangle = \langle 0,0,1\rangle$. Hence

$$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4\,\mathbf{i} - 8\,\mathbf{j}.$$ Thus, parametric equations are: $x = -2 + 4t$, $y = 2 - 8t$, $z = 4$.

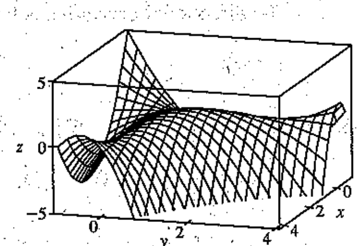
51. $f(x,y) = x^2 - xy + y^2 + 9x - 6y + 10 \Rightarrow f_x = 2x - y + 9$, $f_y = -x + 2y - 6$, $f_{xx} = 2 = f_{yy}$, $f_{xy} = -1$. Then $f_x = 0$ and $f_y = 0$ imply $y = 1$, $x = -4$. Thus the only critical point is $(-4,1)$ and $f_{xx}(-4,1) > 0$, $D(-4,1) = 3 > 0$, so $f(-4,1) = -11$ is a local minimum.



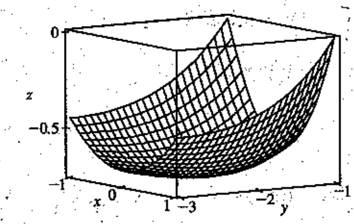
52. $f(x,y) = x^3 - 6xy + 8y^3 \Rightarrow f_x = 3x^2 - 6y$, $f_y = -6x + 24y^2$, $f_{xx} = 6x$, $f_{yy} = 48y$, $f_{xy} = -6$. Then $f_x = 0$ implies $y = x^2/2$, substituting into $f_y = 0$ implies $6x(x^3 - 1) = 0$, so the critical points are $(0,0)$, $(1, \frac{1}{2})$. $D(0,0) = -36 < 0$ so $(0,0)$ is a saddle point while $f_{xx}(1, \frac{1}{2}) = 6 > 0$ and $D(1, \frac{1}{2}) = 108 > 0$ so $f(1, \frac{1}{2}) = -1$ is a local minimum.



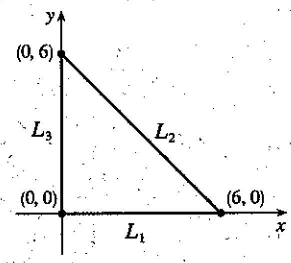
53. $f(x,y) = 3xy - x^2y - xy^2 \Rightarrow f_x = 3y - 2xy - y^2$, $f_y = 3x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 3 - 2x - 2y$. Then $f_x = 0$ implies $y(3 - 2x - y) = 0$ so $y = 0$ or $y = 3 - 2x$. Substituting into $f_y = 0$ implies $x(3 - x) = 0$ or $3x(-1 + x) = 0$. Hence the critical points are $(0,0)$, $(3,0)$, $(0,3)$ and $(1,1)$. $D(0,0) = D(3,0) = D(0,3) = -9 < 0$ so $(0,0)$, $(3,0)$, and $(0,3)$ are saddle points. $D(1,1) = 3 > 0$ and $f_{xx}(1,1) = -2 < 0$, so $f(1,1) = 1$ is a local maximum.



54. $f(x, y) = (x^2 + y)e^{y/2} \Rightarrow f_x = 2xe^{y/2}$, $f_y = e^{y/2}(2 + x^2 + y)/2$, $f_{xx} = 2e^{y/2}$, $f_{yy} = e^{y/2}(4 + x^2 + y)/4$, $f_{xy} = xe^{y/2}$. Then $f_x = 0$ implies $x = 0$, so $f_y = 0$ implies $y = -2$. But $f_{xx}(0, -2) > 0$, $D(0, -2) = e^{-2} - 0 > 0$ so $f(0, -2) = -2/e$ is a local minimum.

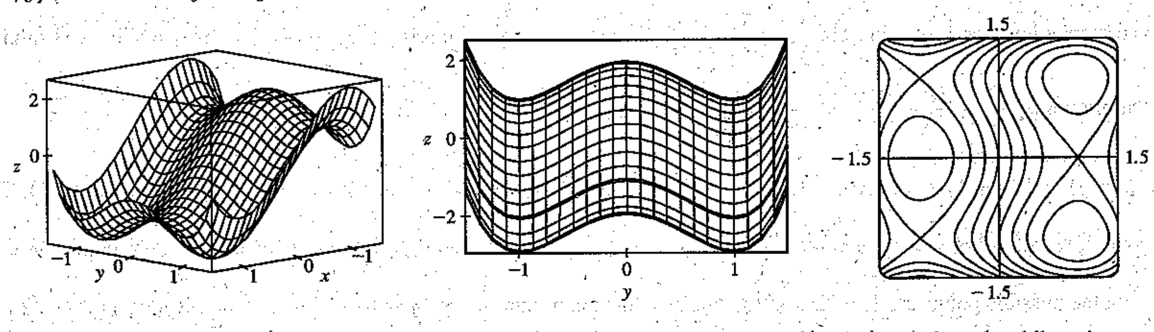


55. First solve inside D. Here $f_x = 4y^2 - 2xy^2 - y^3$, $f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4 - 2x$, but $y = 0$ isn't inside D. Substituting $y = 4 - 2x$ into $f_y = 0$ implies $x = 0$, $x = 2$ or $x = 1$, but $x = 0$ isn't inside D, and when $x = 2$, $y = 0$ but $(2, 0)$ isn't inside D. Thus the only critical point inside D is $(1, 2)$ and $f(1, 2) = 4$. Secondly we consider the boundary of D. On L_1: $f(x, 0) = 0$ and so $f = 0$ on L_1. On L_2: $x = -y + 6$ and $f(-y + 6, y) = y^2(6 - y)(-2) = -2(6y^2 - y^3)$ which has critical points at $y = 0$ and $y = 4$. Then $f(6, 0) = 0$ while $f(2, 4) = -64$. On L_3: $f(0, y) = 0$, so $f = 0$ on L_3. Thus on D the absolute maximum of f is $f(1, 2) = 4$ while the absolute minimum is $f(2, 4) = -64$.



56. Inside D: $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0, 0)$, $(0, \pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0, 0) = 0$, $f(\pm 1, 0) = e^{-1}$ and $f(0, \pm 1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f(x, y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0, \pm 2) = 8e^{-4}$. Thus on D the absolute maximum of f is $f(0, \pm 1) = 2e^{-1}$ and the absolute minimum is $f(0, 0) = 0$.

57. $f(x, y) = x^3 - 3x + y^4 - 2y^2$

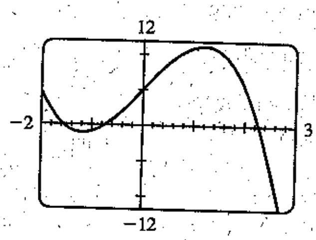


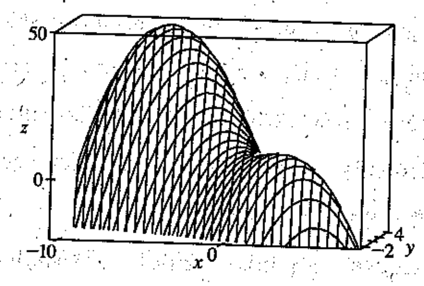
From the graphs, it appears that f has a local maximum $f(-1, 0) \approx 2$, local minima $f(1, \pm 1) \approx -3$, and saddle points at $(-1, \pm 1)$ and $(1, 0)$.

To find the exact quantities, we calculate $f_x = 3x^2 - 3 = 0 \Leftrightarrow x = \pm 1$ and $f_y = 4y^3 - 4y = 0 \Leftrightarrow y = 0, \pm 1$, giving the critical points estimated above. Also $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4$, so using the Second

Derivatives Test, $D(-1,0) = 24 > 0$ and $f_{xx}(-1,0) = -6 < 0$ indicating a local maximum $f(-1,0) = 2$; $D(1,\pm1) = 48 > 0$ and $f_{xx}(1,\pm1) = 6 > 0$ indicating local minima $f(1,\pm1) = -3$; and $D(-1,\pm1) = -48$ and $D(1,0) = -24$, indicating saddle points.

58. $f(x,y) = 12 + 10y - 2x^2 - 8xy - y^4 \Rightarrow f_x(x,y) = -4x - 8y$, $f_y(x,y) = 10 - 8x - 4y^3$. Now $f_x(x,y) = 0 \Rightarrow x = -2x$, and substituting this into $f_y(x,y) = 0$ gives $10 + 16y - 4y^3 = 0 \Leftrightarrow 5 + 8y - 2y^3 = 0$.





From the first graph, we see that this is true when $y \approx -1.542, -0.717$, or 2.260. (Alternatively, we could have found the solutions to $f_x = f_y = 0$ using a CAS.) So to three decimal places, the critical points are $(3.085, -1.542)$, $(1.434, -0.717)$, and $(-4.519, 2.260)$. Now in order to use the Second Derivatives Test, we calculate $f_{xx} = -4$, $f_{xy} = -8$, $f_{yy} = -12y^2$, and $D = 48y^2 - 64$. So since $D(3.085, -1.542) > 0$, $D(1.434, -0.717) < 0$, and $D(-4.519, 2.260) > 0$, and f_{xx} is always negative, $f(x,y)$ has local maxima $f(-4.519, 2.260) \approx 49.373$ and $f(3.085, -1.542) \approx 9.948$, and a saddle point at approximately $(1.434, -0.717)$. The highest point on the graph is approximately $(-4.519, 2.260, 49.373)$.

59. $f(x,y) = x^2y$, $g(x,y) = x^2 + y^2 = 1 \Rightarrow \nabla f = \langle 2xy, x^2 \rangle = \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2xy = 2\lambda x$ implies $x = 0$ or $y = \lambda$. If $x = 0$ then $x^2 + y^2 = 1$ gives $y = \pm1$ and we have possible points $(0, \pm1)$ where $f(0,\pm1) = 0$. If $y = \lambda$ then $x^2 = 2\lambda y$ implies $x^2 = 2y^2$ and substitution into $x^2 + y^2 = 1$ gives $3y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{3}}$ and $x = \pm\sqrt{\frac{2}{3}}$. The corresponding possible points are $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{3}}\right)$. The absolute maximum is $f\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$ while the absolute minimum is $f\left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$.

60. $f(x,y) = 1/x + 1/y$, $g(x,y) = 1/x^2 + 1/y^2 = 1 \Rightarrow \nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$. Then $-x^{-2} = -2\lambda x^3$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm\sqrt{2})$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

61. $f(x,y,z) = xyz$, $g(x,y,z) = x^2 + y^2 + z^2 = 3$. $\nabla f = \lambda \nabla g \Rightarrow \langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$. If any of x, y, or z is zero, then $x = y = z = 0$ which contradicts $x^2 + y^2 + z^2 = 3$. Then $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} = \dfrac{xy}{2z} \Rightarrow 2y^2z = 2x^2z \Rightarrow$

$y^2 = x^2$, and similarly $2yz^2 = 2x^2y \;\Rightarrow\; z^2 = x^2$. Substituting into the constraint equation gives $x^2 + x^2 + x^2 = 3 \;\Rightarrow$ $x^2 = 1 = y^2 = z^2$. Thus the possible points are $(1, 1, \pm 1)$, $(1, -1, \pm 1)$, $(-1, 1, \pm 1)$, $(-1, -1, \pm 1)$. The absolute maximum is $f(1, 1, 1) = f(1, -1, -1) = f(-1, 1, -1) = f(-1, -1, 1) = 1$ and the absolute minimum is $f(1, 1, -1) = f(1, -1, 1) = f(-1, 1, 1) = f(-1, -1, -1) = -1$.

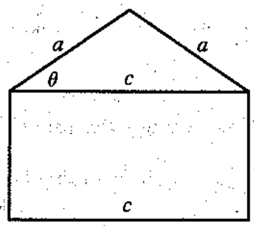
62. $f(x, y, z) = x^2 + 2y^2 + 3z^2$, $g(x, y, z) = x + y + z = 1$, $h(x, y, z) = x - y + 2z = 2 \;\Rightarrow$ $\nabla f = \langle 2x, 4y, 6z \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda + \mu, \lambda - \mu, \lambda + 2\mu \rangle$ and $2x = \lambda + \mu$ **(1)**, $4y = \lambda - \mu$ **(2)**, $6z = \lambda + 2\mu$ **(3)**, $x + y + z = 1$ **(4)**, $x - y + 2z = 2$ **(5)**. Then six times **(1)** plus three times **(2)** plus two times **(3)** implies $12(x + y + z) = 11\lambda + 7\mu$, so **(4)** gives $11\lambda + 7\mu = 12$. Also six times **(1)** minus three times **(2)** plus four times **(3)** implies $12(x - y + 2z) = 7\lambda + 17\mu$, so **(5)** gives $7\lambda + 17\mu = 24$. Solving $11\lambda + 7\mu = 12$, $7\lambda + 17\mu = 24$ simultaneously gives $\lambda = \frac{6}{23}$, $\mu = \frac{30}{23}$. Substituting into **(1)**, **(2)**, and **(3)** implies $x = \frac{18}{23}$, $y = -\frac{6}{23}$, $z = \frac{11}{23}$ giving only one point. Then $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$. Now since $(0, 0, 1)$ satisfies both constraints and $f(0, 0, 1) = 3 > \frac{33}{23}$, $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$ is an absolute minimum, and there is no absolute maximum.

63. $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = xy^2z^3 = 2 \;\Rightarrow\; \nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle \lambda y^2 z^3, 2\lambda xyz^3, 3\lambda xy^2z^2 \rangle$. Since $xy^2z^3 = 2$, $x \neq 0$, $y \neq 0$ and $z \neq 0$, so $2x = \lambda y^2 z^3$ **(1)**, $1 = \lambda xz^3$ **(2)**, $2 = 3\lambda xy^2 z$ **(3)**. Then **(2)** and **(3)** imply $\dfrac{1}{xz^3} = \dfrac{2}{3xy^2z}$ or $y^2 = \frac{2}{3}z^2$ so $y = \pm z\sqrt{\frac{2}{3}}$. Similarly **(1)** and **(3)** imply $\dfrac{2x}{y^2z^3} = \dfrac{2}{3xy^2z}$ or $3x^2 = z^2$ so $x = \pm\frac{1}{\sqrt{3}}z$. But $xy^2z^3 = 2$ so x and z must have the same sign, that is, $x = \frac{1}{\sqrt{3}}z$. Thus $g(x, y, z) = 2$ implies $\frac{1}{\sqrt{3}}z\left(\frac{2}{3}z^2\right)z^3 = 2$ or $z = \pm 3^{1/4}$ and the possible points are $(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4})$, $(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4})$. However at each of these points f takes on the same value, $2\sqrt{3}$. But $(2, 1, 1)$ also satisfies $g(x, y, z) = 2$ and $f(2, 1, 1) = 6 > 2\sqrt{3}$. Thus f has an absolute minimum value of $2\sqrt{3}$ and no absolute maximum subject to the constraint $xy^2z^3 = 2$.

Alternate solution: $g(x, y, z) = xy^2z^3 = 2$ implies $y^2 = \dfrac{2}{xz^3}$, so minimize $f(x, z) = x^2 + \dfrac{2}{xz^3} + z^2$. Then $f_x = 2x - \dfrac{2}{x^2z^3}$, $f_z = -\dfrac{6}{xz^4} + 2z$, $f_{xx} = 2 + \dfrac{4}{x^3z^3}$, $f_{zz} = \dfrac{24}{xz^5} + 2$ and $f_{xz} = \dfrac{6}{x^2z^4}$. Now $f_x = 0$ implies $2x^3z^3 - 2 = 0$ or $z = 1/x$. Substituting into $f_y = 0$ implies $-6x^3 + 2x^{-1} = 0$ or $x = \frac{1}{\sqrt[4]{3}}$, so the two critical points are $\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$. Then $D\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = (2 + 4)\left(2 + \frac{24}{3}\right) - \left(\frac{6}{\sqrt{3}}\right)^2 > 0$ and $f_{xx}\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = 6 > 0$, so each point is a minimum. Finally, $y^2 = \dfrac{2}{xz^3}$, so the four points closest to the origin are $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$.

64. $V = xyz$, say x is the length and $x + 2y + 2z \leq 108$, $x > 0$, $y > 0$, $z > 0$. First maximize V subject to $x + 2y + 2z = 108$ with x, y, z all positive. Then $\langle yz, xz, xy \rangle = \langle \lambda, 2\lambda, 2\lambda \rangle$ implies $2yz = xz$ or $x = 2y$ and $xz = xy$ or $z = y$. Thus $g(x, y, z) = 108$ implies $6y = 108$ or $y = 18 = z$, $x = 36$, so the volume is $V = 11{,}664$ cubic units. Since $(104, 1, 1)$ also satisfies $g(x,y,z) = 108$ and $V(104,1,1) = 104$ cubic units, $(36,18,18)$ gives an absolute maximum of V subject to $g(x,y,z) = 108$. But if $x + 2y + 2z < 108$, there exists $\alpha > 0$ such that $x + 2y + 2z = 108 - \alpha$ and as above $6y = 108 - \alpha$ implies $y = (108-\alpha)/6 = z$, $x = (108-\alpha)/3$ with $V = (108-\alpha)^3/(6^2 \cdot 3) < (108)^3/(6^2 \cdot 3) = 11{,}664$. Hence we have shown that the maximum of V subject to $g(x,y,z) \le 108$ is the maximum of V subject to $g(x,y,z) = 108$ (an intuitively obvious fact).

65.



The area of the triangle is $\frac{1}{2}ca\sin\theta$ and the area of the rectangle is bc. Thus, the area of the whole object is $f(a,b,c) = \frac{1}{2}ca\sin\theta + bc$. The perimeter of the object is $g(a,b,c) = 2a + 2b + c = P$. To simplify $\sin\theta$ in terms of a, b, and c notice that $a^2\sin^2\theta + \left(\frac{1}{2}c\right)^2 = a^2 \;\Rightarrow\; \sin\theta = \frac{1}{2a}\sqrt{4a^2 - c^2}$. Thus $f(a,b,c) = \frac{c}{4}\sqrt{4a^2 - c^2} + bc$. (Instead of using θ, we could just have used the Pythagorean Theorem.) As a result, by Lagrange's method, we must find a, b, c, and λ by solving $\nabla f = \lambda \nabla g$ which gives the following equations: $ca(4a^2 - c^2)^{-1/2} = 2\lambda$ **(1)**, $c = 2\lambda$ **(2)**, $\frac{1}{4}(4a^2 - c^2)^{1/2} - \frac{1}{4}c^2(4a^2 - c^2)^{-1/2} + b = \lambda$ **(3)**, and $2a + 2b + c = P$ **(4)**. From **(2)**, $\lambda = \frac{1}{2}c$ and so **(1)** produces $ca(4a^2 - c^2)^{-1/2} = c \;\Rightarrow\; (4a^2 - c^2)^{1/2} = a \;\Rightarrow\; 4a^2 - c^2 = a^2 \;\Rightarrow\; c = \sqrt{3}\,a$ **(5)**. Similarly, since $(4a^2 - c^2)^{1/2} = a$ and $\lambda = \frac{1}{2}c$, **(3)** gives $\frac{a}{4} - \frac{c^2}{4a} + b = \frac{c}{2}$, so from **(5)**, $\frac{a}{4} - \frac{3a}{4} + b = \frac{\sqrt{3}\,a}{2} \;\Rightarrow\; -\frac{a}{2} - \frac{\sqrt{3}\,a}{2} = -b \;\Rightarrow\; b = \frac{a}{2}(1 + \sqrt{3})$ **(6)**. Substituting **(5)** and **(6)** into **(4)** we get:

$2a + a(1+\sqrt{3}) + \sqrt{3}\,a = P \;\Rightarrow\; 3a + 2\sqrt{3}\,a = P \;\Rightarrow\; a = \frac{P}{3 + 2\sqrt{3}} = \frac{2\sqrt{3} - 3}{3}P$ and thus

$b = \frac{(2\sqrt{3} - 3)(1 + \sqrt{3})}{6}P = \frac{3 - \sqrt{3}}{6}P$ and $c = (2 - \sqrt{3})P$.

66. (a) $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + f(x(t), y(t))\,\mathbf{k} \;\Rightarrow\; \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \left(f_x\frac{dx}{dt} + f_y\frac{dy}{dt}\right)\mathbf{k}$

(by the Chain Rule). Therefore

$$K = \tfrac{1}{2}m\,|\mathbf{v}|^2 = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(f_x\frac{dx}{dt} + f_y\frac{dy}{dt}\right)^2\right]$$

$$= \frac{m}{2}\left[(1 + f_x^2)\left(\frac{dx}{dt}\right)^2 + 2f_xf_y\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + (1 + f_y^2)\left(\frac{dy}{dt}\right)^2\right]$$

(b) $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\,\mathbf{i} + \frac{d^2y}{dt^2}\,\mathbf{j} + \left[f_{xx}\left(\frac{dx}{dt}\right)^2 + 2f_{xy}\frac{dx}{dt}\frac{dy}{dt} + f_{yy}\left(\frac{dy}{dt}\right)^2 + f_x\frac{d^2x}{dt^2} + f_y\frac{d^2y}{dt^2}\right]\mathbf{k}$

(c) If $z = x^2 + y^2$, where $x = t\cos t$ and $y = t\sin t$, then $z = f(x,y) = t^2$.

$\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{v} = (\cos t - t\sin t)\,\mathbf{i} + (\sin t + t\cos t)\,\mathbf{j} + 2t\,\mathbf{k}$,

$K = \frac{m}{2}[(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (2t)^2] = \frac{m}{2}(1 + t^2 + 4t^2) = \frac{m}{2}(1 + 5t^2)$, and

$\mathbf{a} = (-2\sin t - t\cos t)\,\mathbf{i} + (2\cos t - t\sin t)\,\mathbf{j} + 2\,\mathbf{k}$. Notice that it is easier not to use the formulas in (a) and (b).