KEY

Math 225: Exam the First

October 7, 2011

This exam is closed book and closed notes. READ ALL DIRECTIONS CAREFULLY!! Please justify all of your answers. You may use a calculator for arithmetic and scienific functions only (ie, no graphing). You have two hours.

1. Let $x = t^2 + 3t + 1$ and $y = 2t - t^2$.

   (a) Find the values of t for which the above curve has horizontal and vertical tangents.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2-2t}{2t+3}$$

horiz. tangent @ $t=1$

vert. tangent @ $t = -\frac{3}{2}$

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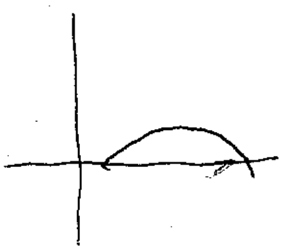
   (b) Find the area bound by this curve and the x–axis. (You'll need to find the t values where the curve meets this axis).

$y=0$

$0 = 2t - t^2$

$\Rightarrow t = 0, 2$

$x = 1, 11$



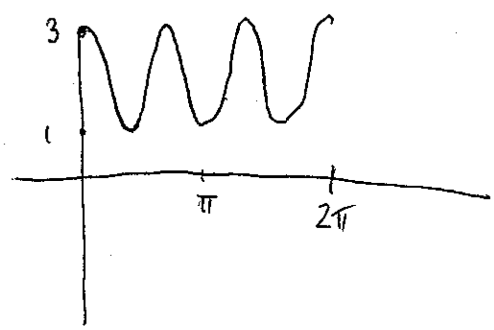
$$\int_{x=1}^{x=11} y\,dx = \int_{t=0}^{t=2} (2t-t^2)(2t+3)\,dt$$

$$= \int_0^2 -2t^3 + t^2 + 6t \; dt = \left[\frac{-2}{4}t^4 + \frac{t^3}{3} + 3t^2\right]_0^2$$

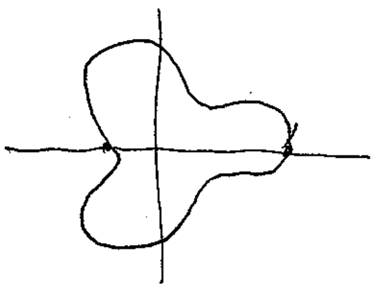
$$-8 + \frac{8}{3} + 12$$

$$= 4 + \frac{8}{3} = \frac{20}{3}$$

2. (a) Draw the graph of $r = 2 + \cos(3\theta)$ in *Cartesian* coordinates.



(b) Use this graph to draw $r = 2 + \cos(3\theta)$ in *polar* coordinates.



(c) Does this graph ever go through the origin? Why or why not?

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No. For this to happen, we'd need $r=0$

but $0 = 2+\cos 3\theta \Rightarrow \cos 3\theta = -2$, which never happens.

(d) Set up, but *do not compute* the integral that computes the area inside of this graph.

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (2+\cos(3\theta))^2 \, d\theta$$

3. Find values x such that the vectors $\langle -5, x, 1\rangle$ and $\langle x, 2x, -3\rangle$ are perpendicular.

$\langle -5, x, 1\rangle \cdot \langle x, 2x, -3\rangle = 0$

$-5x + 2x^2 - 3 = 0$

$2x^2 - 5x - 3 = 0$

$(2x+1)(x-3) = 0 \quad x = 3, -\frac{1}{2}$

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4. (a) Find the point of intersection of the two lines:

$$L_1(t) = \langle 1, 3, 6\rangle + t\langle -1, 1, 2\rangle$$

$$L_2(t) : \frac{x+1}{2} = \frac{-5+y}{-2} = \frac{z+4}{3}$$

$L_1(t) : x = 1 - t \quad = 2s - 1 \quad = x : L_2(s)$

$y = 3 + t \quad = -2s + 5 \quad = y$

$z = 6 + 2t \quad = 3s - 4 \quad = z$

* Solving for x, y leads to a null solution

so solve for y, z instead:

$3 + t = -2s + 5$

$(\times -2)$ $6 + 2t = 3s - 4 \quad s = 2 \quad$ point

$-6 - 2t = +4s - 10 \quad t = -2 \quad (1, 3, 2)$

$7s = 14$

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(b) Find the plane that contains both of these lines.

$\vec{n} = \langle -1, 1, 2\rangle$

$\times \langle 2, -2, 3\rangle$

$\langle 7, 7, 0\rangle$

point: $(1, 3, 6)$

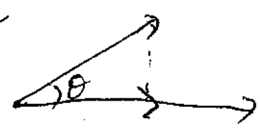
plane: $7(x-1) + 7(y-3) + 0(z-6) = 0$

or $7x + 7y = 28$ or $x + y = 4$.

5. (a) Under what conditions is $\mathbf{proj_a b} = \vec{0}$? Give both a geometric and algebraic justification of your answer.

If $a = 0$, $\text{proj}_{\vec{a}} \vec{b} = \vec{0}$

else



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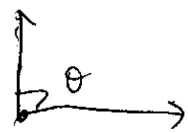

if $\vec{a} \perp \vec{b}$, i.e, $\theta = \frac{\pi}{2}$,

then $\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a} = \vec{0} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$.

(b) Under what conditions is $\mathbf{proj_a b} = \mathbf{b}$? Give both a geometric and algebraic justification of your answer.

$$\vec{b} = \text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

$$|\vec{b}| = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{a}|} = \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{a}|}$$ so $\cos\theta = 1$ or $\cos\theta = -1$

if $\theta = 0$ or $\theta = \pi$

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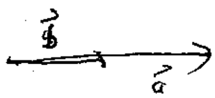
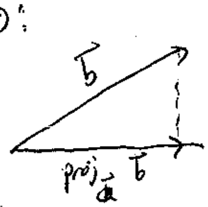


if $\vec{a} \parallel \vec{b}$

(c) Is it ever the case that $|\mathbf{proj_a b}| > |\mathbf{b}|$? Discuss.

NO:



$\text{proj}_{\vec{a}} \vec{b}$ is one leg on the triangle with $\vec{b}$ as the hypotenuse.

6. (a) The curve

$$\mathbf{r}(t) = \langle \cos t, 1 + \cos^2 t, \sin t \rangle$$

is the intersection of which two surfaces? (Give one equation involving x and z, and one equation involving all three of x, y, and z).

NB: Other answers Possible:

$x = \cos t$
$z = \sin t$

$x^2 + z^2 = 1 \leftarrow$ cylinder

$y = 1 + \cos^2 t$
$= \cos^2 t + \sin^2 t + \cos^2 t$
$= x^2 + z^2 + x^2$
$y = 2x^2 + z^2$
paraboloid.

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(b) Give the equation of the tangent line to $\mathbf{r}(t)$ when $t = \pi$.

$\vec{r}(\pi) = \langle \cos\pi, 1+\cos^2\pi, \sin\pi \rangle$
$= \langle -1, 2, 0 \rangle$

$\vec{r}'(t) = \langle -\sin t, -2\cos t \sin t, \cos t \rangle$
$\vec{r}'(\pi) = \langle 0, 0, -1 \rangle$

line: $\langle -1, 2, 0 \rangle + t\langle 0, 0, -1 \rangle$

7. (a) Give the definition for the *unit binormal vector* to a space curve, $\mathbf{r}(t)$, $\mathbf{B}(t)$.

$$\vec{B}(t) = \underbrace{\vec{T}(t)}_{\text{Unit tangent}} \times \underbrace{\vec{N}(t)}_{\text{Unit normal}}$$

(b) Prove that $\mathbf{B}'$ is orthogonal to $\mathbf{B}$. (Use the fact that $\mathbf{B}$ is unit and differentiate).

WTS. $\vec{B}' \perp \vec{B}$.

Using that $|\vec{B}| = 1$ $\quad \vec{B} \circ \vec{B} = 1$

$$(\vec{B} \circ \vec{B})' = 0$$
$$2\vec{B} \circ B' = 0$$
$$\vec{B} \circ \vec{B}' = 0 \Rightarrow \vec{B} \perp B'.$$

(c) Prove that $\mathbf{B}'$ is orthogonal to $\mathbf{T}$. (Use your definition from A and differentiate.

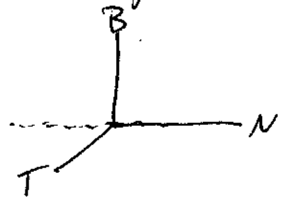
$$\vec{B}' = (\vec{T} \times \vec{N})' = \overbrace{\vec{T}' \times \vec{N}}^{0 \text{ since } T' \parallel N} + \vec{T} \times \vec{N}'$$

so $\vec{B}' = \vec{T} \times \vec{N}'$

$\vec{B}'$ is a vector orthogonal to $\vec{T}$ (and $\vec{N}'$)

(d) Conclude that $\mathbf{B}'$ is parallel to $\mathbf{N}$.

$\vec{B}'$ is orthogonal to both $\vec{T}$ & $\vec{B}$, as is N, so



B' is parallel to $\vec{B} \times \vec{T} = N$.

8. (Bonus) Which mathematical term comes from...

(a) The Latin for 'to kiss'? osculating plane/circle

(b) The Greek for 'to throw'? parabola / hyperbola

(c) The Latin for 'to touch'? tangent

(d) The French for 'snail'? limaçon.