# Odd-numbered Problems

## Practice Exam the Second

1. a) $f(x,y)=\frac{y}{x^2}$   $f=-2,-1,0,1,2$

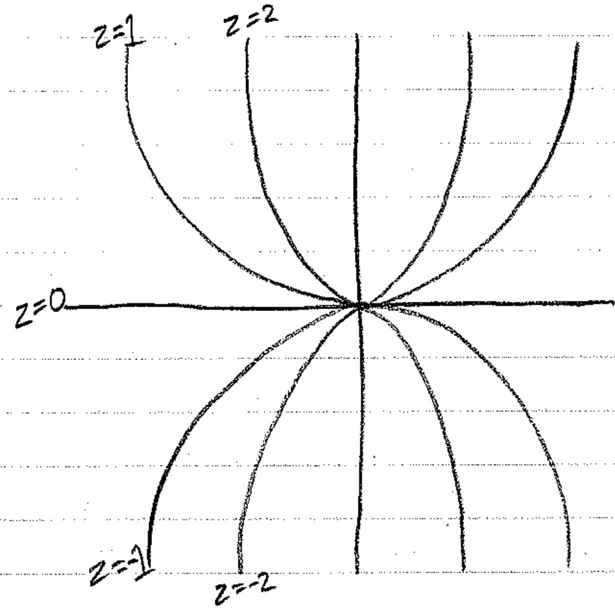
$\frac{y}{x^2}=-2$   $y=-2x^2$

$\frac{y}{x^2}=-1$   $y=-x^2$

$\frac{y}{x^2}=0$   $y=0$

$\frac{y}{x^2}=1$   $y=x^2$

$\frac{y}{x^2}=2$   $y=2x^2$




b) $f(0,0)$ is undefined, so the apparent intersection does not happen.

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3. $\lim\limits_{(x,y)\to(0,0)} \frac{\cos(x^2+y^2)-1}{(x^2+y^2)^2}$   Convert to polar coordinates

$\lim\limits_{r\to 0} \frac{\cos(r^2)-1}{r^4}$   Use L'Hospital's rule

$=\lim\limits_{r\to 0} \frac{-\sin(r^2)2r}{4r^3}$

$=\lim\limits_{r\to 0} \frac{-\sin(r^2)}{2r^2} \to 1$   The limit of $\frac{\sin\theta}{\theta}$ goes to 1

$=\frac{1}{2}$

5. $f(x,y) = x^4y - x^2y^3$

a) Looking for $-\vec{\nabla}f$ at the point $(2,-3)$

$$\vec{\nabla}f = \langle f_x, f_y \rangle$$
$$= \langle 4x^3y - 2xy^3, x^4 - 3x^2y^2 \rangle$$

$$\vec{\nabla}f(2,-3) = \langle -96+108, 16-108 \rangle$$
$$= \langle 12, -92 \rangle$$

Direction of fastest decrease: $\langle -12, 92 \rangle$

b) $z = z_0 + f_x(x-x_0) + f_y(y-y_0)$ Equation for tangent plane
$$= 60 + 12(x-2) - 92(y+3)$$

$f(2,-3) = -48 + 108 = 60$

$$f(2.02, 2.97) \approx 60 + 12(2.02-2) - 92(-2.97+3)$$
$$\approx 60 + 12(.02) - 92(.03)$$
$$\approx 60 + .24 - 2.76$$
$$\approx 60 - 2.52$$
$$\approx 57.48$$

c) $x^4y - x^2y^3 = 60$ 1 degree of freedom.

$\frac{dy}{dx} = \frac{-F_x}{F_y}$ $(F_y \neq 0)$ Implicit Function Theorem

$$= \left. \frac{-(4x^3y - 2xy^3)}{x^4 - 3x^2y^2} \right|_{(2,-3)} = \frac{-12}{-92} = \frac{12}{92}$$

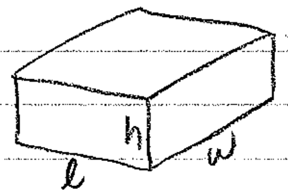
The "slope" of $\vec{\nabla}f = \frac{-92}{12}$, which is the negative reciprocal of $\frac{dy}{dx}$, which is because they are perpendicular.

# Odd-numbered Problems

continued

"Related rates on crack!"
-Balof

7.




$\frac{dh}{dt} = 1$ in/min $\quad \frac{dw}{dt} = -.5$ in/min $\quad \frac{d\ell}{dt} = 2$ in/min

$\ell = 3 \quad w = 4 \quad h = 5$

$V = \ell w h \qquad SA = 2(\ell w + wh + \ell h)$

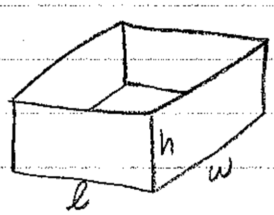
$$\frac{dV}{dt} = \ell w \frac{dh}{dt} + \ell h \frac{dw}{dt} + wh \frac{d\ell}{dt}$$

Calc I: Product Rule used
Calc III: Chain Rule used

$$= (12)(1) + (15)(-.5) + (20)(2)$$
$$= 12 - 7.5 + 40$$
$$= 44.5 \text{ in}^3/\text{min}$$

$$\frac{dS}{dt} = 2(w+\ell)\frac{dh}{dt} + 2(h+\ell)\frac{dw}{dt} + 2(w+h)\frac{d\ell}{dt}$$
$$= (14)(1) + (16)(-.5) + (18)(2)$$
$$= 14 - 8 + 36$$
$$= 42 \text{ in}^2/\text{min}$$

9.




$S = 2wh + 2\ell h + \ell w$ Maximize this subject to $\ell w h = 27$

$\boxed{\text{Method 1: Plug in the constraint}}$

$h = \frac{27}{\ell w} \qquad S = \frac{2w\,27}{\ell w} + \frac{2\ell\,27}{\ell w} + \ell w$

$$= \frac{54}{\ell} + \frac{54}{w} + \ell w$$

$S_\ell = \frac{-54}{\ell^2} + w = 0 \qquad w = \frac{54}{\ell^2}$

$S_w = \frac{-54}{w^2} + \ell = 0 \qquad \ell = \frac{54}{w^2} = \frac{54}{(54/\ell^2)^2} = \frac{\ell^4}{54}$

$\ell = \sqrt[3]{54} \qquad w = \sqrt[3]{54}$

$$h = \frac{27}{(\sqrt[3]{54})^2} = \frac{27}{(\sqrt[3]{27})(\sqrt[3]{2})^2} = \frac{\sqrt[3]{27}}{(\sqrt[3]{2})^2} = \sqrt[3]{\frac{27}{4}} = h$$

These values give us a minimum function value, because the constraint does not keep the possible surface areas from blowing up to infinity.

$\boxed{\text{Method 2: Lagrange Multipliers}}$

$\underbrace{S = 2wh + 2lh + lw}_{f}$ subject to $\underbrace{lwh = 27}_{g}$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$f_w = 2h + l = \lambda lh = \lambda g_w \qquad w(2h+l) = \lambda wlh$

$f_l = 2h + w = \lambda hw = \lambda g_l \qquad l(2h+w) = \lambda wlh$

$f_h = 2w + 2l = \lambda wl = \lambda g_h \qquad h(2w+2l) = \lambda wlh$

$2wh + wl = 2lh + wl = 2wh + 2hl \qquad wl = 2wh$

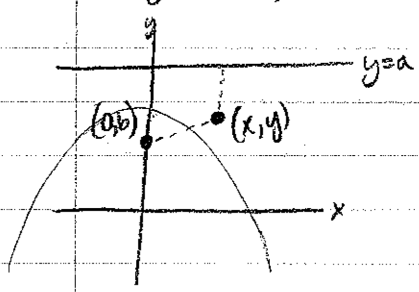
$l = 2h$

$l = w = 2h \qquad l \cdot l \cdot \frac{l}{2} = 27$

$l^3 = 54$

$\boxed{l = \sqrt[3]{54} = w \qquad h = \frac{\sqrt[3]{54}}{2}}$

11. $(x,y) \quad (0,b)$




$D = |y-a|$

$y - a = \sqrt{x^2 + (y-b)^2}$

$(y-a)^2 = x^2 + (y-b)^2$

$\cancel{y^2} - 2ay + a^2 = x^2 + \cancel{y^2} - 2by + b^2$

$2by - 2ay = x^2 + b^2 - a^2$

$y = \frac{x^2 + b^2 - a^2}{2b - 2a} = \frac{1}{2b-2a}x^2 + \left(\frac{b^2 - a^2}{2b-2a}\right)$

This is a parabola

Def. of a parabola is the set of all points a set distance between a point and a line. The point is the focus, and the line is the directrix.

# Practice Exam
## Review    Even Problems

2. $$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2} =$$

$x=y$ $\displaystyle \lim_{x\to 0} \frac{(x-x)^2}{x^2+x^2} = \frac{0}{2x^2} = 0$

$x=-y$ $\displaystyle \lim_{x\to 0} \frac{(x+x)^2}{x^2+x^2} = \lim_{x\to 0} \frac{4x^2}{2x^2} = 2$

limits don't agree. DNE

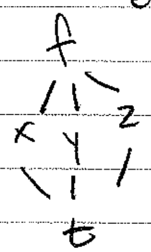
4.

$f_x(x,y) = 2xy + x^3$

$f(x,y) = x^2y + \frac{x^4}{4} + g(y)$

$f_y(x,y) = x^2 + 0 + g'(y)$ ← any variation as long as it follows this format

6. $f(x,y,z) = xy + z^2$   $\langle x,y,z\rangle = \langle t, t^2, t^4\rangle$



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$= (y \cdot 1) + (x \cdot 2t) + (2z \cdot 4t^3)$

$= t^2 + 2t^2 + 8t^7$

$\frac{df}{dt} = 3t^2 + 8t^7$   $t=1$

$$\frac{df}{dt} = 11$$

8. $f(x,y)=\frac{x^3}{3}+\frac{y^3}{3}-\frac{x^2}{2}-y^2$

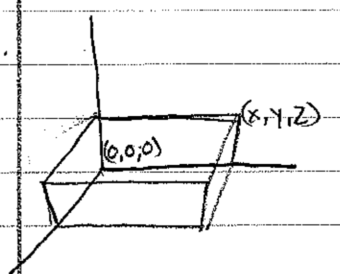
$f_x = x^2 - x = 0 \qquad x=1,\ x=0$

$f_y = y^2 - 2y = 0 \qquad y=0,\ y=2$

| $(0,0)$ | $(0,2)$ | $(1,0)$ | $(1,2)$ |
|---|---|---|---|
| $D>0$ | $D<0$ | $D<0$ | $D>0$ |
| max or min | saddle | saddle | max or min |

$D=(f_{xx})(f_{yy})-(f_{xy})^2$

$D=(2x-1)(2y-2)-0$

10. 


$V=xyz$ subject to $x+2y+3z=10$

$V_x = yz = \lambda \qquad xz = 2yz$

$V_y = xz = 2\lambda \qquad x=2y$

$V_z = xy = 3\lambda$

$x=3z$

$x+x+x=10$

$3x=10 \quad x=10/3 \quad y=10/6 \quad z=10/9$

12.
a) $\int_0^2\int_0^4 yx^2+xy+y^2\,dy\,dx \qquad [0,2]\times[0,4]$

$\qquad\qquad x \qquad y$

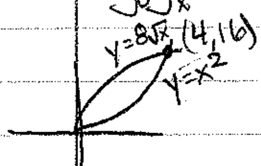
Now calculate!

b) $\int_0^1\int_0^2 y\cos(xy)\,dy\,dx \xrightarrow{\text{Flip}} \int_0^2\int_0^1 y\cos(xy)\,dx\,dy$

Now Calculate!

* use u-sub!

c) $\int_0^4\int_{x^2}^{8\sqrt{x}} x+2y\,dy\,dx = \int_0^{16}\int_{\frac{y^2}{64}}^{\sqrt{y}} x+2y\,dx\,dy$




Now Calculate!