Math 225: Practice Final

These practice problems come from the final in Fall 2009. They are meant to represent the types of problems that you're likely to encounter on the final. Note that this set of problems is not exhaustive. Any inclusions or omissions here should not induce you to study more or fewer topics.

1. Consider the four points $A = (1, 2, 1)$, $B = (3, 4, 5)$, $C(2, -1, 6)$, and $D = (4, 1, 10)$.
   (a) Prove that $ABCD$ is a parallelogram by proving that AB is parallel to CD and AC is parallel to BD.
   (b) Find the area of parallelogram $ABCD$.
   (c) Find angle BAC (you can leave your answer as an arccosine).
2. Consider the following parametric equation
   $$\mathbf{r}(t) = \langle t, \sqrt{9 - t^2}, t^2 \rangle$$
   (a) For which values of t is the curve defined?
   (b) Find the tangent line when $t = 1$
   (c) Can we find the vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ when $t = 3$? Why or why not?
   (d) Set up the integral to compute the length of $\mathbf{r}(t)$ from $t = -1$ to $t = 1$.
3. Consider $f(x, y) = 2x + y^2$
   (a) What do the level curves of f look like?
   (b) Find the tangent plane at (3, 4).
   (c) What is $D_{\mathbf{u}}f$ at (3,4), where $\mathbf{u}$ moves from $(3, 4)$ to $(5, 5)$?
4. Recall from the second exam that a *harmonic* function is a function f that satisfies
   $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$
   (a) If f and g are harmonic, is their sum, $f + g$, harmonic?
   (b) If f and g are harmonic, is their product, fg, harmonic?
5. Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to $(x-1)^2 + (y-1)^2 = 4$, and explain why your answer makes geometric sense.
6. Solve the following integral by reversing the order of integration.
   $$\int_0^1 \int_0^{\arcsin(y)} y \, dx \, dy$$
7. Find the surface area of the plane $ax + by - z = h$ inside of the cylinder $x^2 + y^2 = 2x$.

8. Justify the following formula: The area between two polar curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, as θ ranges from α to β, is given by

$$Area = \frac{1}{2}\int_{\alpha}^{\beta} (f_2(\theta))^2 - (f_1(\theta))^2 \, d\theta$$

9. (a) Find, using calculus, the volume of a sphere of radius R.

   (b) Based on your answer to (a), conjecture a formula for the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

10. Find the following line integrals over the given curves.

   (a)

$$\int_C (x+y)\, dx + (x + e^y)\, dy$$

   where C is the parabolic segment along $y = x^2 + 2$ from (1,3) to (3, 11).

   (b)

$$\int_C (2x+y)\, dx + 2y\, dy$$

   where C is the line segment from $(1,3)$ to $(0,0)$

   (c)

$$\oint_C (\cos(x) + 2y)\, dx + (x^2 + \sin(y))\, dy$$

   where C is the triangle from $(0,0)$ to $(3,0)$ to $(3,1)$.

11. List as many (at least 3) integral theorems as you can and draw any parallels that you can between them. Note: I want *theorems* and not *formulas* (ie, 'The integral for arclength' is not a *theorem*).

12. Choose a concept from single-variable calculus that was revisited in multivariable calculus and explain the parallels and the differences between the concept in the two contexts. Write your answer to someone who is finishing Calculus II and is thinking about taking Calculus III.