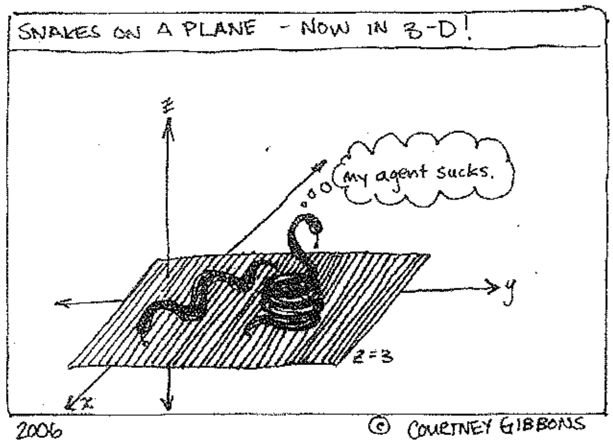
KEY

Math 225: Quiz the Second
September 16, 2011

This quiz is closed book and closed notes. You may not use a calculator on this quiz. Please justify all of your answers. You have until 5 minutes before the hour to finish.

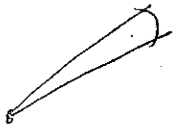


1. (a) Give the formula for the slope of the tangent line to a polar curve $r = f(\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

(b) When looking at an area in polar coordinates, rather than dividing it up into tiny rectangles, we divide it up into what shapes?

6

 tiny circle sectors

(c) Find a unit vector in the same direction as $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

$$|\vec{i} - \vec{j} + 3\vec{k}| = \sqrt{1^2+1^2+9^2} = \sqrt{11}$$

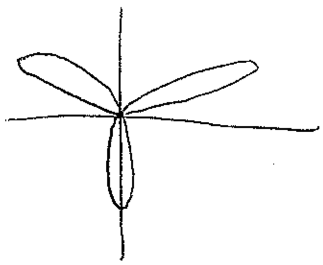
so $\vec{u} = \left\langle \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}} \right\rangle$ works.

8

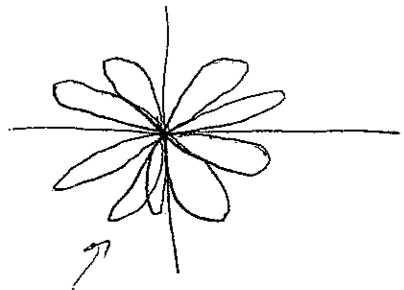
2. Recall that the curve $r = \sin(n\theta)$ is an $n-$leaved rose if n is odd, and a $2n$-leaved rose if n is even. Find the total area bound by the leaves in $r = \sin(n\theta)$ for both n even and n odd. (Hint: Take the area of one leaf and multiply by the appropriate number. You should only have to work one integral). For a bonus, comment on what your answer is missing.

Eg

$n = 3$



$n = 4$



NB: I shouldn't do these in pen!

$$\text{one leaf} = \frac{1}{2}\int_0^{\pi/n} (\sin(n\theta))^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/n} \left(\frac{1-\cos(2n\theta)}{2}\right) d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 2n\theta}{2n}\right]_0^{\pi/n}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{n} - 0\right) - (0-0)\right] = \frac{\pi}{4n}$$

$$\left.\begin{array}{l}\text{odd } n: \ \text{Area} = n\left(\frac{\pi}{4n}\right) = \frac{\pi}{4} \\ \text{Even } n: \ \text{Area} = 2n\left(\frac{\pi}{4n}\right) = \pi/2\end{array}\right\}$$

Amazingly, both answers are independent of n!

3. Give a geometric description of the intersection of the surfaces $x + y = 1$ and $y + z = 1$? Also, give at least two points on this intersection.

(5)

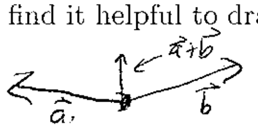
$x + y = 1$
$y + z = 1$ are planes so their intersection is a line

a point on that line: if $x = 0$ then $y = 1$ and $z = 0$ $\rightarrow (0, 1, 0)$

if $x = 1$, $y = 0$ and $z = 1$ $\rightarrow (1, 0, 1)$

4. True or False? (Justify if true, give a counterexample if false)

(a) $|\mathbf{a} + \mathbf{b}| \geq |\mathbf{a}| + |\mathbf{b}|$
(b) $|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| + |\mathbf{b}|$
(You will find it helpful to draw some examples here).

(a) 

$\vec{a} + \vec{b}$ is much shorter than $\vec{a}$ or $\vec{b}$, so (a) is false

(6) (b) 

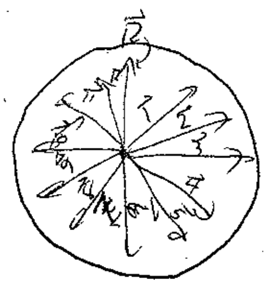
$\vec{a} - \vec{b}$ is much shorter than $\vec{a}$ or $\vec{b}$, so (b) is false.

5. (Bonus)

(a) The face of a clock has 12 vectors, each with their tail at the center, and each with their head on a different number. What is the sum of these vectors.

(b) The face of the clock has 11 vectors, each with their tail on '12', and each pointing to a different number. What is the sum of these vectors?

a)



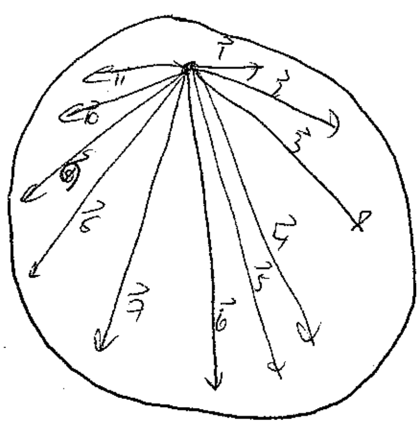
$``\vec{12}" + ``\vec{6}" = \vec{0}$

$``\vec{1}" + ``\vec{7}" = \vec{0}$

etc

so $\sum_{i=1}^{12} ``\vec{i}" = \vec{0}$

b)



$``\vec{1}" + ``\vec{7}" = ``\vec{6}"$

$``\vec{2}" + ``8" = \vec{6}"$

$``\vec{3}" + ``\vec{9}" = ``\vec{6}"$

$``\vec{4}" + ``\vec{10}" = ``\vec{6}"$

$``\vec{5}" + ``\vec{11}" = ``\vec{6}"$

$+ \ \vec{6}$

$\sum_{i=1}^{11} ``\vec{i}" \quad 6 \cdot \vec{6}$